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# Chapter 4
# Fluid Kinematics

## Introductory Problems

### 4-1C

**Solution** We are to define and explain kinematics and fluid kinematics.

***Analysis*** Kinematics means the study of *motion*. Fluid kinematics is the study of how fluids flow and how to describe fluid motion. Fluid kinematics deals with describing the motion of fluids without considering (or even understanding) the forces and moments that *cause* the motion.

***Discussion*** Fluid kinematics deals with such things as describing how a fluid particle translates, distorts, and rotates, and how to visualize flow fields.

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### 4-2

**Solution** We are to write an equation for centerline speed through a nozzle, given that the flow speed increases parabolically.

***Assumptions*** **1** The flow is steady. **2** The flow is axisymmetric. **3** The water is incompressible.

***Analysis*** A general equation for a parabola in the $x$ direction is

*General parabolic equation*:

$$u = a + b(x - c)^2 \tag{1}$$

We have two boundary conditions, namely at $x = 0$, $u = u_{\text{entrance}}$ and at $x = L$, $u = u_{\text{exit}}$. By inspection, Eq. 1 is satisfied by setting $c = 0$, $a = u_{\text{entrance}}$ and $b = (u_{\text{exit}} - u_{\text{entrance}})/L^2$. Thus, Eq. 1 becomes

*Parabolic speed*:

$$u = u_{\text{entrance}} + \frac{(u_{\text{exit}} - u_{\text{entrance}})}{L^2}x^2 \tag{2}$$

***Discussion*** You can verify Eq. 2 by plugging in $x = 0$ and $x = L$.

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## 4-3

**Solution** For a given velocity field we are to find out if there is a stagnation point. If so, we are to calculate its location.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The velocity field is

$$\vec{V} = (u, v) = (0.5 + 1.2x)\vec{i} + (-2.0 - 1.2y)\vec{j} \qquad (1)$$

At a stagnation point, both $u$ and $v$ must equal zero. At any point $(x,y)$ in the flow field, the velocity components $u$ and $v$ are obtained from Eq. 1,

*Velocity components*:

$$u = 0.5 + 1.2x \qquad v = -2.0 - 1.2y \qquad (2)$$

Setting these to zero yields

*Stagnation point*:

$$\begin{aligned} 0 &= 0.5 + 1.2x & x &= -0.4167 \\ 0 &= -2.0 - 1.2y & y &= -1.667 \end{aligned} \qquad (3)$$

So, *yes there is a stagnation point*; its location is $\boldsymbol{x = -0.417, y = -1.67}$ (to 3 digits).

***Discussion*** If the flow were three-dimensional, we would have to set $w = 0$ as well to determine the location of the stagnation point. In some flow fields there is more than one stagnation point.

## 4-4

**Solution** For a given velocity field we are to find out if there is a stagnation point. If so, we are to calculate its location.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x-y* plane.

***Analysis*** The velocity field is

$$\vec{V} = (u, v) = \left(a^2 - (b - cx)^2\right)\vec{i} + \left(-2cby + 2c^2xy\right)\vec{j} \qquad (1)$$

At a stagnation point, both $u$ and $v$ must equal zero. At any point ($x$,$y$) in the flow field, the velocity components $u$ and $v$ are obtained from Eq. 1,

*Velocity components*: $$u = a^2 - (b - cx)^2 \qquad v = -2cby + 2c^2xy \qquad (2)$$

Setting these to zero and solving simultaneously yields

*Stagnation point*: $$\begin{aligned} 0 &= a^2 - (b - cx)^2 & x &= \frac{b-a}{c} \\ v &= -2cby + 2c^2xy & y &= 0 \end{aligned} \qquad (3)$$

So, *yes there is a stagnation point*; its location is $\boldsymbol{x = (b - a)/c, y = 0}$.

***Discussion*** If the flow were three-dimensional, we would have to set $w$ = 0 as well to determine the location of the stagnation point. In some flow fields there is more than one stagnation point.

# Lagrangian and Eulerian Descriptions

## 4-5C

**Solution** We are to define the Lagrangian description of fluid motion.

***Analysis*** In the Lagrangian description of fluid motion, individual fluid particles (fluid elements composed of a fixed, identifiable mass of fluid) are followed.

***Discussion*** The Lagrangian method of studying fluid motion is similar to that of studying billiard balls and other solid objects in physics.

**4-6C**

**Solution** We are to compare the Lagrangian method to the study of systems and control volumes and determine to which of these it is most similar.

***Analysis*** The Lagrangian method is more similar to **system analysis**. In both cases, we follow a mass of fixed identity as it moves in a flow. In a control volume analysis, on the other hand, mass moves into and out of the control volume, and we don't follow any particular chunk of fluid. Instead we analyze whatever fluid happens to be inside the control volume at the time.

***Discussion*** In fact, the Lagrangian analysis is the same as a system analysis in the limit as the size of the system shrinks to a point.

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**4-7C**

**Solution** We are to define the Eulerian description of fluid motion, and explain how it differs from the Lagrangian description.

***Analysis*** In the Eulerian description of fluid motion, we are concerned with *field variables*, such as velocity, pressure, temperature, etc., as functions of space and time within a flow domain or control volume. In contrast to the Lagrangian method, fluid flows into and out of the Eulerian flow domain, and we do not keep track of the motion of particular identifiable fluid particles.

***Discussion*** The Eulerian method of studying fluid motion is not as "natural" as the Lagrangian method since the fundamental conservation laws apply to moving particles, not to fields.

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**4-8C**

**Solution** We are to determine whether a measurement is Lagrangian or Eulerian.

***Analysis*** Since the probe is fixed in space and the fluid flows around it, we are *not* following individual fluid particles as they move. Instead, we are measuring a field variable at a particular location in space. Thus this is an **Eulerian** measurement.

***Discussion*** If a neutrally buoyant probe were to move with the flow, its results would be Lagrangian measurements – following fluid particles.

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## 4-9C

**Solution** We are to determine whether a measurement is Lagrangian or Eulerian.

***Analysis*** Since the probe moves with the flow and is neutrally buoyant, we are following individual fluid particles as they move through the pump. Thus this is a **Lagrangian** measurement.

***Discussion*** If the probe were instead fixed at one location in the flow, its results would be Eulerian measurements.

## 4-10C

**Solution** We are to determine whether a measurement is Lagrangian or Eulerian.

***Analysis*** Since the weather balloon moves with the air and is neutrally buoyant, we are following individual "fluid particles" as they move through the atmosphere. Thus this is a **Lagrangian** measurement. Note that in this case the "fluid particle" is huge, and can follow gross features of the flow – the balloon obviously cannot follow small scale turbulent fluctuations in the atmosphere.

***Discussion*** When weather monitoring instruments are mounted on the roof of a building, the results are Eulerian measurements.

## 4-11C

**Solution** We are to determine whether a measurement is Lagrangian or Eulerian.

***Analysis*** Relative to the airplane, the probe is fixed and the air flows around it. We are *not* following individual fluid particles as they move. Instead, we are measuring a field variable at a particular location in space relative to the moving airplane. Thus this is an **Eulerian** measurement.

***Discussion*** The airplane is moving, but it is not moving with the flow.

**4-12C**

**Solution** We are to compare the Eulerian method to the study of systems and control volumes and determine to which of these it is most similar.

***Analysis*** The Eulerian method is more similar to **control volume analysis**. In both cases, mass moves into and out of the flow domain or control volume, and we don't follow any particular chunk of fluid. Instead we analyze whatever fluid happens to be inside the control volume at the time.

***Discussion*** In fact, the Eulerian analysis is the same as a control volume analysis except that Eulerian analysis is usually applied to infinitesimal volumes and differential equations of fluid flow, whereas control volume analysis usually refers to finite volumes and integral equations of fluid flow.

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**4-13C**

**Solution** We are to define a steady flow field in the Eulerian description, and discuss particle acceleration in such a flow.

***Analysis*** A flow field is defined as steady in the Eulerian frame of reference when **properties at any point in the flow field do not change with respect to time**. In such a flow field, individual fluid particles may still experience non-zero acceleration – the answer to the question is **yes**.

***Discussion*** Although velocity is not a function of time in a steady flow field, its total derivative with respect to time $\left(\vec{a} = d\vec{V} / dt\right)$ is not necessarily zero since the acceleration is composed of a local (unsteady) part which is zero and an advective part which is not necessarily zero.

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**4-14C**

**Solution** We are to list three alternate names for material derivative.

***Analysis*** The material derivative is also called **total derivative**, **particle derivative**, **Eulerian derivative**, **Lagrangian derivative,** and **substantial derivative**. "Total" is appropriate because the material derivative includes both local (unsteady) and convective parts. "Particle" is appropriate because it stresses that the material derivative is one following fluid particles as they move about in the flow field. "Eulerian" is appropriate since the material derivative is used to transform from Lagrangian to Eulerian reference frames. "Lagrangian" is appropriate since the material derivative is used to transform from Lagrangian to Eulerian reference frames. Finally, "substantial" is not as clear of a term for the material derivative, and we are not sure of its origin.

***Discussion*** All of these names emphasize that we are following a fluid particle as it moves through a flow field.

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## 4-15

**Solution** We are to calculate the material acceleration for a given velocity field.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The velocity field is

$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

The acceleration field components are obtained from its definition (the material acceleration) in Cartesian coordinates,

*Material acceleration components*:

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 + (U_0 + bx)b + (-by)0 + 0 \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = 0 + (U_0 + bx)0 + (-by)(-b) + 0 \end{aligned} \qquad (2)$$

where the unsteady terms are zero since this is a steady flow, and the terms with *w* are zero since the flow is two-dimensional. Eq. 2 simplifies to

*Material acceleration components*: $$a_x = b(U_0 + bx) \qquad a_y = b^2 y \qquad (3)$$

In terms of a vector,

*Material acceleration vector*: $$\vec{a} = b(U_0 + bx)\vec{i} + b^2 y\vec{j} \qquad (4)$$

***Discussion*** For positive *x* and *b*, fluid particles accelerate in the positive *x* direction. Even though this flow is steady, there is still a non-zero acceleration field.

## 4-16

**Solution** For a given pressure and velocity field, we are to calculate the rate of change of pressure following a fluid particle.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* plane.

***Analysis*** The pressure field is

*Pressure field*:
$$P = P_0 - \frac{\rho}{2}\left[2U_0 bx + b^2\left(x^2 + y^2\right)\right] \qquad (1)$$

By definition, the material derivative, when applied to pressure, produces the rate of change of pressure following a fluid particle. Using Eq. 1 and the velocity components from Problem 4-15,

*Rate of change of pressure following a fluid particle*:
$$\frac{DP}{Dt} = \underbrace{\cancel{\frac{\partial P}{\partial t}}}_{\text{Steady}} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} + \underbrace{\cancel{w\frac{\partial P}{\partial z}}}_{\text{Two-dimensional}} \qquad (2)$$
$$= \left(U_0 + bx\right)\left(-\rho U_0 b - \rho b^2 x\right) + \left(-by\right)\left(-\rho b^2 y\right)$$

where the unsteady term is zero since this is a steady flow, and the term with *w* is zero since the flow is two-dimensional. Eq. 2 simplifies to

*Rate of change of pressure following a fluid particle*:
$$\boxed{\frac{DP}{Dt} = \rho\left[-U_0^{\,2} b - 2U_0 b^2 x + b^3\left(y^2 - x^2\right)\right]} \qquad (3)$$

***Discussion*** The material derivative can be applied to any flow property, scalar or vector. Here we apply it to the pressure, a scalar quantity.

**4-17**

**Solution** For a given velocity field we are to calculate the acceleration.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The velocity components are

*Velocity components*: $u = 1.1 + 2.8x + 0.65y \qquad v = 0.98 - 2.1x - 2.8y$ (1)

The acceleration field components are obtained from its definition (the material acceleration) in Cartesian coordinates,

*Acceleration components*:

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ &= 0 + (1.1 + 2.8x + 0.65y)(2.8) + (0.98 - 2.1x - 2.8y)(0.65) + 0 \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ &= 0 + (1.1 + 2.8x + 0.65y)(-2.1) + (0.98 - 2.1x - 2.8y)(-2.8) + 0 \end{aligned} \tag{3}$$

where the unsteady terms are zero since this is a steady flow, and the terms with *w* are zero since the flow is two-dimensional. Eq. 3 simplifies to

*Acceleration components*: $a_x = 3.717 + 6.475x \qquad a_y = -5.054 + 6.475y$ (4)

At the point $(x,y) = (-2,3)$, the acceleration components of Eq. 4 are

*Acceleration components at* (-2,3): $a_x = \mathbf{-9.233} \qquad a_y = \mathbf{14.371}$

***Discussion*** No units are given in either the problem statement or the answers. We assume that the coefficients have appropriate units.

## 4-18

**Solution** For a given velocity field we are to calculate the acceleration.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The velocity components are

*Velocity components*: $u = 0.20 + 1.3x + 0.85y \qquad v = -0.50 + 0.95x - 1.3y$ (1)

The acceleration field components are obtained from its definition (the material acceleration) in Cartesian coordinates,

*Acceleration components*:

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ &= 0 + \left(0.20 + 1.3x + 0.85y\right)\left(1.3\right) + \left(-0.50 + 0.95x - 1.3y\right)\left(0.85\right) + 0 \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ &= 0 + \left(0.20 + 1.3x + 0.85y\right)\left(0.95\right) + \left(-0.50 + 0.95x - 1.3y\right)\left(-1.3\right) + 0 \end{aligned} \tag{3}$$

where the unsteady terms are zero since this is a steady flow, and the terms with *w* are zero since the flow is two-dimensional. Eq. 3 simplifies to

*Acceleration components*: $a_x = -0.165 + 2.4975x \qquad a_y = 0.84 + 2.4975y$ (4)

At the point (*x*,*y*) = (1,2), the acceleration components of Eq. 4 are

*Acceleration components at* (1,2): $a_x = \mathbf{2.3325} \qquad a_y = \mathbf{5.835}$

***Discussion*** No units are given in either the problem statement or the answers. We assume that the coefficients have appropriate units.

**4-19**

**Solution** We are to generate an expression for the fluid acceleration for a given velocity.

***Assumptions*** **1** The flow is steady. **2** The flow is axisymmetric. **3** The water is incompressible.

***Analysis*** In Problem 4-2 we found that along the centerline,

*Speed along centerline of nozzle*:
$$u = u_{\text{entrance}} + \frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x^2 \qquad (1)$$

To find the acceleration in the *x*-direction, we use the material acceleration,

*Acceleration along centerline of nozzle*:
$$a_x = \cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} \qquad (2)$$

The first term in Eq. 2 is zero because the flow is steady. The last two terms are zero because the flow is axisymmetric, which means that along the centerline there can be no *v* or *w* velocity component. We substitute Eq. 1 for *u* to obtain

*Acceleration along centerline of nozzle*:
$$a_x = u\frac{\partial u}{\partial x} = \left(u_{\text{entrance}} + \frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x^2\right)(2)\frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x \qquad (3)$$

or

$$\boxed{a_x = 2u_{\text{entrance}}\frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x + 2\frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)^2}{L^4}x^3} \qquad (4)$$

***Discussion*** Fluid particles are accelerated along the centerline of the nozzle, even though the flow is steady.

**4-20**

**Solution** We are to write an equation for centerline speed through a diffuser, given that the flow speed decreases parabolically.

***Assumptions*** **1** The flow is steady. **2** The flow is axisymmetric.

***Analysis*** A general equation for a parabola in $x$ is

*General parabolic equation*:

$$u = a + b\left(x - c\right)^2 \qquad (1)$$

We have two boundary conditions, namely at $x = 0$, $u = u_{\text{entrance}}$ and at $x = L$, $u = u_{\text{exit}}$. By inspection, Eq. 1 is satisfied by setting $c = 0$, $a = u_{\text{entrance}}$ and $b = (u_{\text{exit}} - u_{\text{entrance}})/L^2$. Thus, Eq. 1 becomes

*Parabolic speed*:

$$u = u_{\text{entrance}} + \frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x^2 \qquad (2)$$

***Discussion*** You can verify Eq. 2 by plugging in $x = 0$ and $x = L$.

**4-21**

**Solution** We are to generate an expression for the fluid acceleration for a given velocity, and then calculate its value at two $x$ locations.

***Assumptions*** **1** The flow is steady. **2** The flow is axisymmetric.

***Analysis*** In Problem 4-20 we found that along the centerline,

*Speed along centerline of diffuser*:
$$u = u_{\text{entrance}} + \frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x^2 \qquad (1)$$

To find the acceleration in the $x$-direction, we use the material acceleration,

*Acceleration along centerline of diffuser*:
$$a_x = \cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} \qquad (2)$$

The first term in Eq. 2 is zero because the flow is steady. The last two terms are zero because the flow is axisymmetric, which means that along the centerline there can be no $v$ or $w$ velocity component. We substitute Eq. 1 for $u$ to obtain

*Acceleration along centerline of diffuser*:
$$a_x = u\frac{\partial u}{\partial x} = \left(u_{\text{entrance}} + \frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x^2\right)(2)\frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x$$

or

$$\boxed{a_x = 2u_{\text{entrance}}\frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)}{L^2}x + 2\frac{\left(u_{\text{exit}} - u_{\text{entrance}}\right)^2}{L^4}x^3} \qquad (3)$$

At the given locations, we substitute the given values. At $x = 0$,

*Acceleration along centerline of diffuser at x = 0*:
$$a_x\left(x = 0\right) = \mathbf{0} \qquad (4)$$

At $x = 1.0$ m,

*Acceleration along centerline of diffuser at x = 1.0 m*:
$$a_x\left(x = 1.0\text{ m}\right) = 2\left(30.0\text{ m/s}\right)\frac{\left(-25.0\text{ m/s}\right)}{\left(2.0\text{ m}\right)^2}\left(1.0\text{ m}\right) + 2\frac{\left(-25.0\text{ m/s}\right)^2}{\left(2.0\text{ m}\right)^4}\left(1.0\text{ m}\right)^3 \qquad (5)$$
$$= \mathbf{-297\ m/s^2}$$

***Discussion*** $a_x$ is negative implying that fluid particles are *decelerated* along the centerline of the diffuser, even though the flow is steady. Because of the parabolic nature of the velocity field, the acceleration is zero at the entrance of the diffuser, but its magnitude increases rapidly downstream.

## Flow Patterns and Flow Visualization

### 4-22C

**Solution** We are to define streamline and discuss what streamlines indicate.

***Analysis*** **A streamline is a curve that is everywhere tangent to the instantaneous local velocity vector**. It indicates the instantaneous direction of fluid motion throughout the flow field.

***Discussion*** If a flow field is steady, streamlines, pathlines, and streaklines are identical.

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### 4-23

**Solution** For a given velocity field we are to generate an equation for the streamlines.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The steady, two-dimensional velocity field of Problem 4-15 is

*Velocity field*:
$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

For two-dimensional flow in the *x*-*y* plane, streamlines are given by

*Streamlines in the x-y plane*:
$$\left.\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u} \qquad (2)$$

We substitute the $u$ and $v$ components of Eq. 1 into Eq. 2 and rearrange to get

$$\frac{dy}{dx} = \frac{-by}{U_0 + bx}$$

We solve the above differential equation by separation of variables:

$$-\int \frac{dy}{by} = \int \frac{dx}{U_0 + bx}$$

Integration yields

$$-\frac{1}{b}\ln(by) = \frac{1}{b}\ln(U_0 + bx) + \frac{1}{b}\ln C_1 \qquad (3)$$

where we have set the constant of integration as the natural logarithm of some constant $C_1$, with a constant in front in order to simplify the algebra (notice that the factor of $1/b$ can be removed from each term in Eq. 3). When we recall that $\ln(ab) = \ln a + \ln b$, and that $-\ln a = \ln(1/a)$, Eq. 3 simplifies to

*Equation for streamlines*:
$$\boxed{y = \frac{C}{(U_0 + bx)}} \qquad (4)$$

The new constant $C$ is related to $C_1$, and is introduced for simplicity.

***Discussion*** Each value of constant $C$ yields a unique streamline of the flow.

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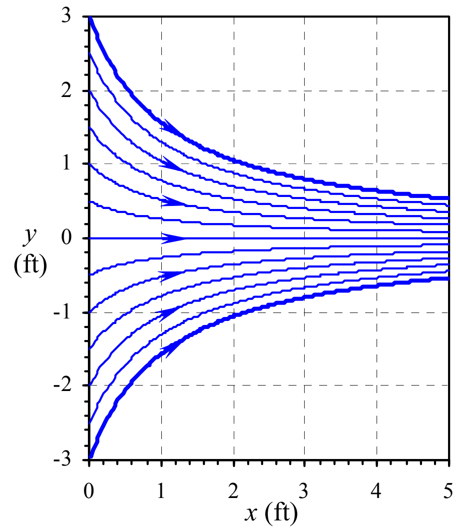


**FIGURE 1**
Streamlines (solid blue curves) for the given velocity field; $x$ and $y$ are in units of ft.

## 4-24E

**Solution** For a given velocity field we are to plot several streamlines for a given range of $x$ and $y$ values.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** From the solution to the previous problem, an equation for the streamlines is

*Streamlines in the x-y plane*:
$$y = \frac{C}{\left(U_0 + bx\right)} \qquad (1)$$

Constant $C$ is set to various values in order to plot the streamlines. Several streamlines in the given range of $x$ and $y$ are plotted in **Fig. 1**.

The direction of the flow is found by calculating $u$ and $v$ at some point in the flow field. We choose $x$ = 1 ft, $y$ = 1 ft. At this point $u$ = 9.6 ft/s and $v$ = –4.6 ft/s. The direction of the velocity at this point is obviously to the lower right. This sets the direction of all the streamlines. The arrows in Fig. 1 indicate the direction of flow.

***Discussion*** The flow is converging channel flow, as illustrated in Fig. P4-15.

## 4-25C

**Solution** We are to determine what kind of flow visualization is seen in a photograph.

***Analysis*** Since the picture is a snapshot of dye streaks in water, each streak shows the time history of dye that was introduced earlier from a port in the body. Thus these are **streaklines**. Since the flow appears to be steady, these streaklines are the same as pathlines and streamlines.

***Discussion*** It is assumed that the dye follows the flow of the water. If the dye is of nearly the same density as the water, this is a reasonable assumption.

## 4-26C

**Solution** We are to define pathline and discuss what pathlines indicate.

***Analysis*** **A pathline is the actual path traveled by an individual fluid particle over some time period**. It indicates the exact route along which a fluid particle travels from its starting point to its ending point. Unlike streamlines, pathlines are not instantaneous, but involve a finite time period.

***Discussion*** If a flow field is steady, streamlines, pathlines, and streaklines are identical.

### 4-27C

**Solution** We are to define streakline and discuss the difference between streaklines and streamlines.

***Analysis*** **A streakline is the locus of fluid particles that have passed sequentially through a prescribed point in the flow**. Streaklines are very different than streamlines. Streamlines are instantaneous curves, everywhere tangent to the local velocity, while streaklines are produced over a finite time period. In an unsteady flow, streaklines distort and then retain features of that distorted shape even as the flow field changes, whereas streamlines change instantaneously with the flow field.

***Discussion*** If a flow field is steady, streamlines and streaklines are identical.

### 4-28C

**Solution** We are to determine what kind of flow visualization is seen in a photograph.

***Analysis*** Since the picture is a snapshot of dye streaks in water, each streak shows the time history of dye that was introduced earlier from a port in the body. Thus these are **streaklines**. Since the flow appears to be unsteady, these streaklines are *not* the same as pathlines or streamlines.

***Discussion*** It is assumed that the dye follows the flow of the water. If the dye is of nearly the same density as the water, this is a reasonable assumption.

### 4-29C

**Solution** We are to determine what kind of flow visualization is seen in a photograph.

***Analysis*** Since the picture is a snapshot of smoke streaks in air, each streak shows the time history of smoke that was introduced earlier from the smoke wire. Thus these are **streaklines**. Since the flow appears to be unsteady, these streaklines are *not* the same as pathlines or streamlines.

***Discussion*** It is assumed that the smoke follows the flow of the air. If the smoke is neutrally buoyant, this is a reasonable assumption. In actuality, the smoke rises a bit since it is hot; however, the air speeds are high enough that this effect is negligible.

## 4-30C

**Solution** We are to determine what kind of flow visualization is seen in a photograph.

***Analysis*** Since the picture is a time exposure of air bubbles in water, each white streak shows the path of an individual air bubble. Thus these are **pathlines**. Since the outer flow (top and bottom portions of the photograph) appears to be steady, these pathlines are the same as streaklines and streamlines.

***Discussion*** It is assumed that the air bubbles follow the flow of the water. If the bubbles are small enough, this is a reasonable assumption.

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## 4-31C

**Solution** We are to define timeline and discuss how timelines can be produced in a water channel. We are also to describe an application where timelines are more useful than streaklines.

***Analysis*** **A timeline is a set of adjacent fluid particles that were marked at the same instant of time**. Timelines can be produced in a water flow by using a hydrogen bubble wire. There are also techniques in which a chemical reaction is initiated by applying current to the wire, changing the fluid color along the wire. Timelines are more useful than streaklines when the uniformity of a flow is to be visualized. Another application is to visualize the velocity profile of a boundary layer or a channel flow.

***Discussion*** Timelines differ from streamlines, streaklines, and pathlines even if the flow is steady.

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## 4-32C

**Solution** For each case we are to decide whether a vector plot or contour plot is most appropriate, and we are to explain our choice.

***Analysis*** In general, contour plots are most appropriate for scalars, while vector plots are necessary when vectors are to be visualized.

(*a*) A **contour plot** of speed is most appropriate since fluid speed is a scalar.
(*b*) A **vector plot** of velocity vectors would clearly show where the flow separates. Alternatively, a vorticity contour plot of vorticity normal to the plane would also show the separation region clearly.
(*c*) A **contour plot** of temperature is most appropriate since temperature is a scalar.
(*d*) A **contour plot** of this component of vorticity is most appropriate since one component of a vector is a scalar.

***Discussion*** There are other options for case (b) – temperature contours can also sometimes be used to identify a separation zone.

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**CD-EES 4-33**

**Solution** For a given velocity field we are to generate an equation for the streamlines and sketch several streamlines in the first quadrant.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x-y* plane.

***Analysis*** The velocity field is given by

$$\vec{V} = (u, v) = (0.5 + 1.2x)\vec{i} + (-2.0 - 1.2y)\vec{j} \qquad (1)$$

For two-dimensional flow in the *x-y* plane, streamlines are given by

*Streamlines in the x-y plane*:
$$\left.\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u} \qquad (2)$$

We substitute the $u$ and $v$ components of Eq. 1 into Eq. 2 and rearrange to get

$$\frac{dy}{dx} = \frac{-2.0 - 1.2y}{0.5 + 1.2x}$$

We solve the above differential equation by separation of variables:

$$\frac{dy}{-2.0 - 1.2y} = \frac{dx}{0.5 + 1.2x} \quad \rightarrow \quad \int \frac{dy}{-2.0 - 1.2y} = \int \frac{dx}{0.5 + 1.2x}$$

Integration yields

$$-\frac{1}{1.2}\ln(-2.0 - 1.2y) = \frac{1}{1.2}\ln(0.5 + 1.2x) - \frac{1}{1.2}\ln C_1 \qquad (3)$$

where we have set the constant of integration as the natural logarithm of some constant $C_1$, with a constant in front in order to simplify the algebra. When we recall that $\ln(ab) = \ln a + \ln b$, and that $-\ln a = \ln(1/a)$, Eq. 3 simplifies to

*Equation for streamlines*:
$$\boxed{y = \frac{C}{1.2(0.5 + 1.2x)} - 1.667}$$

The new constant $C$ is related to $C_1$, and is introduced for simplicity. $C$ can be set to various values in order to plot the streamlines. Several streamlines in the upper right quadrant of the given flow field are shown in Fig. 1.

The direction of the flow is found by calculating $u$ and $v$ at some point in the flow field. We choose $x = 3$, $y = 3$. At this point $u = 4.1$ and $v = -5.6$. The direction of the velocity at this point is obviously to the lower right. This sets the direction of all the streamlines. The arrows in Fig. 1 indicate the direction of flow.

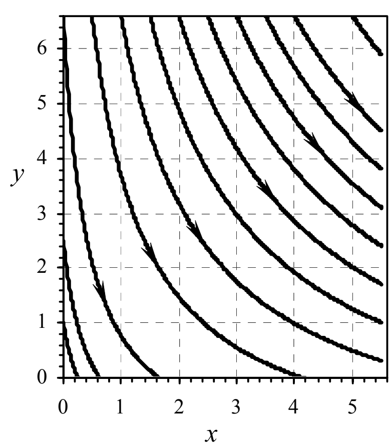


**FIGURE 1**
Streamlines (solid black curves) for the given velocity field.

***Discussion*** The flow appears to be a counterclockwise turning flow in the upper right quadrant.

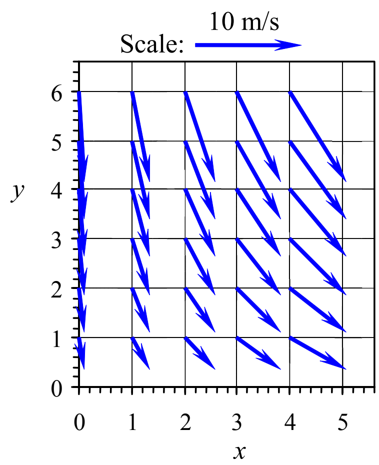


**FIGURE 1**
Velocity vectors for the given velocity field. The scale is shown by the top arrow.

## 4-34

**Solution** For a given velocity field we are to generate a velocity vector plot in the first quadrant.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The velocity field is given by

$$\vec{V} = (u, v) = (0.5 + 1.2x)\vec{i} + (-2.0 - 1.2y)\vec{j} \qquad (1)$$

At any point (*x*,*y*) in the flow field, the velocity components *u* and *v* are obtained from Eq. 1,

*Velocity components*: $u = 0.5 + 1.2x \qquad v = -2.0 - 1.2y \qquad (2)$

To plot velocity vectors, we simply pick an (*x*,*y*) point, calculate *u* and *v* from Eq. 2, and plot an arrow with its tail at (*x*,*y*), and its tip at (*x*+*Su*,*y*+*Sv*) where *S* is some scale factor for the vector plot. For the vector plot shown in Fig. 1, we chose *S* = 0.2, and plot velocity vectors at several locations in the first quadrant.

***Discussion*** The flow appears to be a counterclockwise turning flow in the upper right quadrant.

## 4-35

**Solution** For a given velocity field we are to generate an acceleration vector plot in the first quadrant.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x-y* plane.

***Analysis*** The velocity field is given by

$$\vec{V} = (u, v) = (0.5 + 1.2x)\vec{i} + (-2.0 - 1.2y)\vec{j} \qquad (1)$$

At any point (*x*,*y*) in the flow field, the velocity components *u* and *v* are obtained from Eq. 1,

*Velocity components*: $$u = 0.5 + 1.2x \qquad v = -2.0 - 1.2y \qquad (2)$$

The acceleration field is obtained from its definition (the material acceleration),

*Acceleration components*:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 + (0.5 + 1.2x)(1.2) + 0 + 0$$
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = 0 + 0 + (-2.0 - 1.2y)(-1.2) + 0 \qquad (3)$$

where the unsteady terms are zero since this is a steady flow, and the terms with *w* are zero since the flow is two-dimensional. Eq. 3 simplifies to

*Acceleration components*: $$a_x = 0.6 + 1.44x \qquad a_y = 2.4 + 1.44y \qquad (4)$$

To plot the acceleration vectors, we simply pick an (*x*,*y*) point, calculate $a_x$ and $a_y$ from Eq. 4, and plot an arrow with its tail at (*x*,*y*), and its tip at ($x+Sa_x$,$y+Sa_y$) where *S* is some scale factor for the vector plot. For the vector plot shown in Fig. 1, we chose *S* = 0.15, and plot acceleration vectors at several locations in the first quadrant.

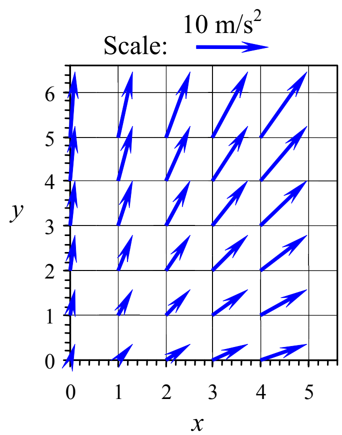


**FIGURE 1**
Acceleration vectors for the velocity field. The scale is shown by the top arrow.

***Discussion*** Since the flow is a counterclockwise turning flow in the upper right quadrant, the acceleration vectors point to the upper right (centripetal acceleration).

## 4-36

**Solution** For the given velocity field, the location(s) of stagnation point(s) are to be determined. Several velocity vectors are to be sketched and the velocity field is to be described.

***Assumptions*** **1** The flow is steady and incompressible. **2** The flow is two-dimensional, implying no $z$-component of velocity and no variation of $u$ or $v$ with $z$.

***Analysis*** ($a$) The velocity field is

$$\vec{V} = (u, v) = (1 + 2.5x + y)\vec{i} + (-0.5 - 1.5x - 2.5y)\vec{j} \qquad (1)$$

Since $\vec{V}$ is a vector, all its components must equal zero in order for $\vec{V}$ itself to be zero. Setting each component of Eq. 1 to zero,

*Simultaneous equations*:
$$u = 1 + 2.5x + y = 0$$
$$v = -0.5 - 1.5x - 2.5y = 0$$

We can easily solve this set of two equations and two unknowns simultaneously. Yes, there is one stagnation point, and it is located at

*Stagnation point*: $x = \mathbf{-0.421\ m}$ $\quad y = \mathbf{0.0526\ m}$

($b$) The $x$ and $y$ components of velocity are calculated from Eq. 1 for several ($x$,$y$) locations in the specified range. For example, at the point ($x$ = 2 m, $y$ = 3 m), $u$ = 9.00 m/s and $v$ = -11 m/s. The magnitude of velocity (the *speed*) at that point is 14.21 m/s. At this and at an array of other locations, the velocity vector is constructed from its two components, the results of which are shown in Fig. 1. The flow can be described as a counterclockwise turning, accelerating flow from the upper left to the lower right. The stagnation point of Part ($a$) does not lie in the upper right quadrant, and therefore does not appear on the sketch.

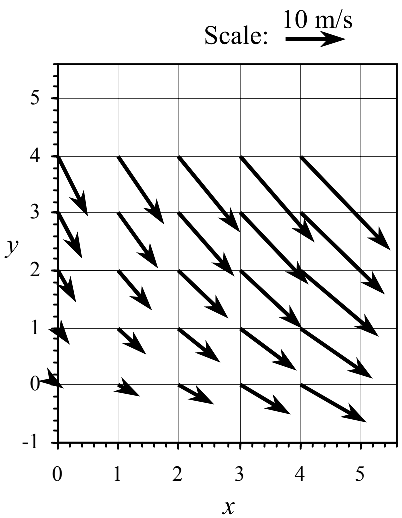


**FIGURE 1**
Velocity vectors in the upper right quadrant for the given velocity field.

***Discussion*** The stagnation point location is given to three significant digits. It will be verified in Chap. 9 that this flow field is physically valid because it satisfies the differential equation for conservation of mass.

## 4-37

**Solution** For the given velocity field, the material acceleration is to be calculated at a particular point and plotted at several locations in the upper right quadrant.

***Assumptions*** **1** The flow is steady and incompressible. **2** The flow is two-dimensional, implying no $z$-component of velocity and no variation of $u$ or $v$ with $z$.

***Analysis*** ($a$) The velocity field is

$$\vec{V} = (u, v) = (1 + 2.5x + y)\vec{i} + (-0.5 - 1.5x - 2.5y)\vec{j} \qquad (1)$$

Using the velocity field of Eq. 1 and the equation for material acceleration in Cartesian coordinates, we write expressions for the two non-zero components of the acceleration vector:

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ &= 0 + (1 + 2.5x + y)(2.5) + (-0.5 - 1.5x - 2.5y)(1) + 0 \end{aligned}$$

and

$$\begin{aligned} a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ &= 0 + (1 + 2.5x + y)(-1.5) + (-0.5 - 1.5x - 2.5y)(-2.5) + 0 \end{aligned}$$

At ($x$ = 2 m, $y$ = 3 m), $a_x$ = **11.5 m/s$^2$** and $a_y$ = **14.0 m/s$^2$**.

($b$) The above equations are applied to an array of $x$ and $y$ values in the upper right quadrant, and the acceleration vectors are plotted in Fig. 1.

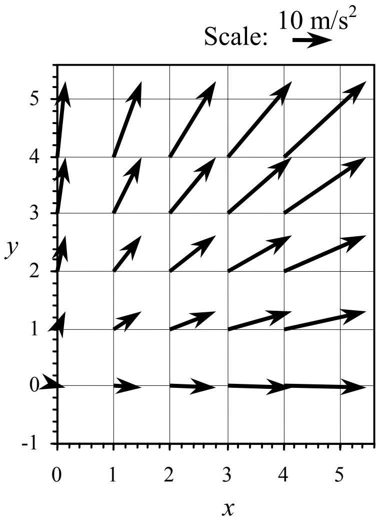


**FIGURE 1**
Acceleration vectors in the upper right quadrant for the given velocity field.

***Discussion*** The acceleration vectors plotted in Fig. 1 point to the upper right, increasing in magnitude away from the origin. This agrees qualitatively with the velocity vectors of Fig. 1 of Problem 4-36; namely, fluid particles are accelerated to the right and are turned in the counterclockwise direction due to centripetal acceleration towards the upper right. Note that the acceleration field is non-zero, even though the flow is steady.

**4-38**

**Solution** For a given velocity field we are to plot a velocity magnitude contour plot at five given values of speed.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis*** Since $u_r = 0$, and since $\omega$ is positive, the speed is equal to the magnitude of the $\theta$-component of velocity,

*Speed*:
$$V = \sqrt{\underbrace{\cancel{u_r^2}}_{0} + u_\theta^2} = |u_\theta| = \omega r$$

Thus, contour lines of constant speed are simply circles of constant radius given by

*Contour line of constant speed*:
$$r = \frac{V}{\omega}$$

For example, at $V$ = 2.0 m/s, the corresponding contour line is a circle of radius 2.0 m,

*Contour line at constant speed V* = 2.0 *m/s*:
$$r = \frac{2.0\text{ m/s}}{1.0\text{ 1/s}} = 2.0\text{ m}$$

We plot a circle at a radius of 2.0 m and repeat this simple calculation for the four other values of $V$. We plot the contours in Fig. 1. The speed increases linearly from the center of rotation (the origin).

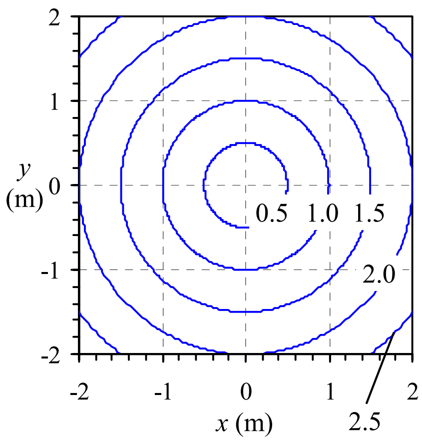


**FIGURE 1**
Contour plot of velocity magnitude for solid body rotation. Values of speed are labeled in units of m/s.

***Discussion*** The contours are equidistant apart because of the linear nature of the velocity field.

## 4-39

**Solution** For a given velocity field we are to plot a velocity magnitude contour plot at five given values of speed.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis*** Since $u_r$ = 0, and since $K$ is positive, the speed is equal to the magnitude of the $\theta$-component of velocity,

*Speed*:
$$V = \sqrt{\underbrace{\cancel{u_r^2}}_{0} + u_\theta^2} = |u_\theta| = \frac{K}{r}$$

Thus, contour lines of constant speed are simply circles of constant radius given by

*Contour line of constant speed*:
$$r = \frac{K}{V}$$

For example, at $V$ = 2.0 m/s, the corresponding contour line is a circle of radius 0.50 m,

*Contour line at constant speed V = 2.0 m/s*:
$$r = \frac{1.0 \text{ m}^2/\text{s}}{2.0 \text{ m/s}} = 0.50 \text{ m}$$

We plot a circle at a radius of 0.50 m and repeat this simple calculation for the four other values of $V$. We plot the contours in Fig. 1. The speed near the center is faster than that further away from the center.

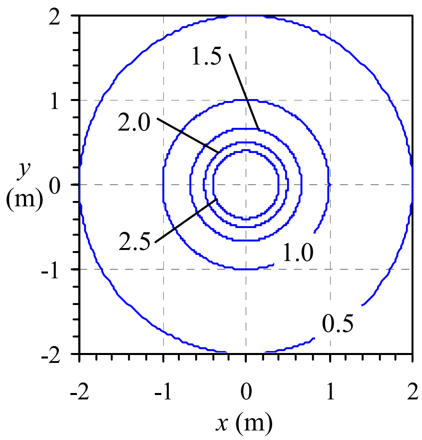


**FIGURE 1**
Contour plot of velocity magnitude for a line vortex. Values of speed are labeled in units of m/s.

***Discussion*** The contours are *not* equidistant apart because of the nonlinear nature of the velocity field.

## 4-40

**Solution** For a given velocity field we are to plot a velocity magnitude contour plot at five given values of speed.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis*** The velocity field is

*Line source*:
$$u_r = \frac{m}{2\pi r} \qquad u_\theta = 0 \tag{1}$$

Since $u_\theta = 0$, and since $m$ is positive, the speed is equal to the magnitude of the $r$-component of velocity,

*Speed*:
$$V = \sqrt{u_r^{\,2} + \underbrace{\cancel{u_\theta^{\,2}}}_{0}} = |u_r| = \frac{m}{2\pi r} \tag{2}$$

Thus, contour lines of constant speed are simply circles of constant radius given by

*Contour line of constant speed*:
$$r = \frac{m}{2\pi V} = \frac{\left(\frac{m}{2\pi}\right)}{V} \tag{3}$$

For example, at $V$ = 2.0 m/s, the corresponding contour line is a circle of radius 0.50 m,

*Contour line at constant speed $V$ = 2.0 m/s*:
$$r = \frac{1.0 \text{ m}^2/\text{s}}{2.0 \text{ m/s}} = 0.50 \text{ m} \tag{4}$$

We plot a circle at a radius of 0.50 m and repeat this simple calculation for the four other values of $V$. We plot the contours in Fig. 1. The flow slows down as it travels further from the origin.

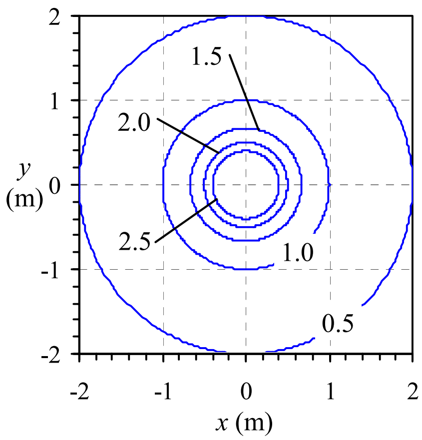


**FIGURE 1**
Contour plot of velocity magnitude for a line source. Values of speed are labeled in units of m/s.

***Discussion*** The contours are *not* equidistant apart because of the nonlinear nature of the velocity field.

## Motion and Deformation of Fluid Elements

### 4-41C

**Solution** We are to name and describe the four fundamental types of motion or deformation of fluid particles.

***Analysis***

1. **Translation** – a fluid particle moves from one location to another.
2. **Rotation** – a fluid particle rotates about an axis drawn through the particle.
3. **Linear strain or extensional strain** – a fluid particle stretches in a direction such that a line segment in that direction is elongated at some later time.
4. **Shear strain** – a fluid particle distorts in such a way that two lines through the fluid particle that are initially perpendicular are *not* perpendicular at some later time.

***Discussion*** In a complex fluid flow, all four of these occur simultaneously.

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### 4-42

**Solution** For a given velocity field, we are to determine whether the flow is rotational or irrotational.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* plane.

***Analysis*** The velocity field is

$$\vec{V} = (u,v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

By definition, the flow is rotational if the vorticity is non-zero. So, we calculate the vorticity. In a 2-D flow in the *x-y* plane, the only non-zero component of vorticity is in the *z* direction, i.e. $\zeta_z$,

*Vorticity component in the z direction*:

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - 0 = 0 \qquad (1)$$

Since the vorticity is zero, this flow is **irrotational**.

***Discussion*** We shall see in Chap. 10 that the fluid very close to the walls is rotational due to important viscous effects near the wall (a *boundary layer*). However, in the majority of the flow field, the irrotational approximation is reasonable.

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**4-43**

**Solution** For a given velocity field we are to generate an equation for the $x$ location of a fluid particle along the $x$-axis as a function of time.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** The velocity field is

*Velocity field*: $$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

We start with the definition of $u$ following a fluid particle,

*x-component of velocity of a fluid particle*: $$\frac{dx_{\text{particle}}}{dt} = u = U_0 + bx_{\text{particle}} \qquad (2)$$

where we have substituted $u$ from Eq. 1. We rearrange and separate variables, dropping the "particle" subscript for convenience,

$$\frac{dx}{U_0 + bx} = dt \qquad (3)$$

Integration yields

$$\frac{1}{b}\ln(U_0 + bx) = t - \frac{1}{b}\ln C_1 \qquad (4)$$

where we have set the constant of integration as the natural logarithm of some constant $C_1$, with a constant in front in order to simplify the algebra. When we recall that $\ln(ab) = \ln a + \ln b$, Eq. 4 simplifies to

$$\ln\left(C_1\left(U_0 + bx\right)\right) = t$$

from which

$$U_0 + bx = C_2 e^{bt} \qquad (5)$$

where $C_2$ is a new constant defined for convenience. We now plug in the known initial condition that at $t = 0$, $x = x_A$ to find constant $C_2$ in Eq. 5. After some algebra,

*Fluid particle's x location at time t*: $$x = x_{A'} = \frac{1}{b}\left[\left(U_0 + bx_A\right)e^{bt} - U_0\right] \qquad (6)$$

***Discussion*** We verify that at $t = 0$, $x = x_A$ in Eq. 6.

**4-44**

**Solution** For a given velocity field we are to generate an equation for the change in length of a line segment moving with the flow along the $x$-axis.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** Using the results of Problem 4-43,

*Location of particle A at time t*:
$$x_{A'} = \frac{1}{b}\left[\left(U_0 + bx_B\right)e^{bt} - U_0\right] \qquad (1)$$

and

*Location of particle B at time t*:
$$x_{B'} = \frac{1}{b}\left[\left(U_0 + bx_B\right)e^{bt} - U_0\right] \qquad (2)$$

Since length $\xi = x_B - x_A$ and length $\xi + \Delta\xi = x_{B'} - x_{A'}$, we write an expression for $\Delta\xi$,

*Change in length of the line segment*:
$$\begin{aligned}\Delta\xi &= \left(x_{B'} - x_{A'}\right) - \left(x_B - x_A\right)\\ &= \frac{1}{b}\left[\left(U_0 + bx_B\right)e^{bt} - U_0\right] - \frac{1}{b}\left[\left(U_0 + bx_A\right)e^{bt} - U_0\right] - \left(x_B - x_A\right)\\ &= x_B e^{bt} - x_A e^{bt} - x_B + x_A\end{aligned} \qquad (3)$$

Eq. 3 simplifies to

*Change in length of the line segment*:
$$\boxed{\Delta\xi = \left(x_B - x_A\right)\left(e^{bt} - 1\right)} \qquad (4)$$

***Discussion*** We verify from Eq. 4 that when $t = 0$, $\Delta\xi = 0$.

**4-45**

**Solution** By examining the increase in length of a line segment along the axis of a converging duct, we are to generate an equation for linear strain rate in the $x$ direction and compare to the exact equation given in this chapter.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** From Problem 4-44we have an expression for the change in length of the line segment AB,

*Change in length of the line segment*: $$\Delta\xi = \left(x_{\mathrm{B}} - x_{\mathrm{A}}\right)\left(e^{bt} - 1\right) \qquad (1)$$

The fundamental definition of linear strain rate is the rate of increase in length of a line segment per unit length of the line segment. For the case at hand,

*Linear strain rate in x direction*: $$\varepsilon_{xx} = \frac{d}{dt}\frac{\left(\xi + \Delta\xi\right) - \xi}{\xi} = \frac{d}{dt}\frac{\Delta\xi}{\xi} = \frac{d}{dt}\frac{\Delta\xi}{x_{\mathrm{B}} - x_{\mathrm{A}}} \qquad (2)$$

We substitute Eq. 1 into Eq. 2 to obtain

*Linear strain rate in x direction*: $$\varepsilon_{xx} = \frac{d}{dt}\frac{\left(x_{\mathrm{B}} - x_{\mathrm{A}}\right)\left(e^{bt} - 1\right)}{x_{\mathrm{B}} - x_{\mathrm{A}}} = \frac{d}{dt}\left(e^{bt} - 1\right) \qquad (3)$$

In the limit as $t \to 0$, we apply the first two terms of the series expansion for $e^{bt}$,

*Series expansion for* $e^{bt}$: $$e^{bt} = 1 + bt + \frac{\left(bt\right)^2}{2!} + \ldots \approx 1 + bt \qquad (4)$$

Finally, for small $t$ we approximate the time derivative as $1/t$, yielding

*Linear strain rate in x direction*: $$\boxed{\varepsilon_{xx} \to \frac{1}{t}\left(1 + bt - 1\right) = b} \qquad (5)$$

Comparing to the equation for $\varepsilon_{xx}$,

*Linear strain rate in x direction*: $$\boxed{\varepsilon_{xx} = \frac{\partial u}{\partial x} = b} \qquad (6)$$

Equations 5 and 6 agree, verifying our algebra.

***Discussion*** Although we considered a line segment on the $x$-axis, it turns out that $\varepsilon_{xx} = b$ everywhere in the flow, as seen from Eq. 6. We could also have taken the analytical time derivative of Eq. 3, yielding $\varepsilon_{xx} = be^{bt}$. Then, as $t \to 0$, $\varepsilon_{xx} \to b$.

## 4-46

**Solution** For a given velocity field we are to generate an equation for the $y$ location of a fluid particle as a function of time.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** The velocity field is

*Velocity field*: $$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

We start with the definition of $v$ following a fluid particle,

*y-component of velocity of a fluid particle*: $$\frac{dy_{\text{particle}}}{dt} = v = -by_{\text{particle}} \qquad (2)$$

where we have substituted $v$ from Eq. 1. We and rearrange and separate variables, dropping the "particle" subscript for convenience,

$$\frac{dy}{y} = -bdt \qquad (3)$$

Integration yields

$$\ln(y) = -bt - \ln C_1 \qquad (4)$$

where we have set the constant of integration as the natural logarithm of some constant $C_1$, with a constant in front in order to simplify the algebra. When we recall that $\ln(ab) = \ln a + \ln b$, Eq. 4 simplifies to

$$\ln(C_1 y) = -t$$

from which

$$y = C_2 e^{-bt} \qquad (5)$$

where $C_2$ is a new constant defined for convenience. We now plug in the known initial condition that at $t = 0$, $y = y_A$ to find constant $C_2$ in Eq. 5. After some algebra,

*Fluid particle's y location at time t*: $$y = y_{A'} = y_A e^{-bt} \qquad (6)$$

***Discussion*** The fluid particle approaches the $x$-axis exponentially with time. The fluid particle also moves downstream in the $x$ direction during this time period. However, in this particular problem $v$ is not a function of $x$, so the streamwise movement is irrelevant ($u$ and $v$ act independently of each other).

**4-47**

**Solution** For a given velocity field we are to generate an equation for the change in length of a line segment in the $y$ direction.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** Using the results of Problem 4-46,

*Location of particle A at time t*: $$y_{A'} = y_A e^{-bt} \quad (1)$$

and

*Location of particle B at time t*: $$y_{B'} = y_B e^{-bt} \quad (2)$$

Since length $\eta = y_B - y_A$ and length $\eta + \Delta\eta = y_{B'} - y_{A'}$, we write an expression for $\Delta\eta$,

*Change in length of the line segment*:

$$\Delta\eta = \left(y_{B'} - y_{A'}\right) - \left(y_B - y_A\right) = y_B e^{-bt} - y_A e^{-bt} - \left(y_B - y_A\right) = y_B e^{-bt} - y_A e^{-bt} - y_B + y_A$$

which simplifies to

*Change in length of the line segment*: $$\boxed{\Delta\eta = \left(y_B - y_A\right)\left(e^{-bt} - 1\right)} \quad (3)$$

***Discussion*** We verify from Eq. 3 that when $t = 0$, $\Delta\eta = 0$.

**4-48**

**Solution** By examining the increase in length of a line segment as it moves down a converging duct, we are to generate an equation for linear strain rate in the $y$ direction and compare to the exact equation given in this chapter.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** From the previous problem we have an expression for the change in length of the line segment AB,

*Change in length of the line segment*:
$$\Delta\eta = \left(y_{\mathrm{B}} - y_{\mathrm{A}}\right)\left(e^{-bt} - 1\right) \qquad (1)$$

The fundamental definition of linear strain rate is the rate of increase in length of a line segment per unit length of the line segment. For the case at hand,

*Linear strain rate in y direction*:
$$\varepsilon_{yy} = \frac{d}{dt}\frac{\left(\eta + \Delta\eta\right) - \eta}{\eta} = \frac{d}{dt}\frac{\Delta\eta}{\eta} = \frac{d}{dt}\frac{\Delta\eta}{y_{\mathrm{B}} - y_{\mathrm{A}}} \qquad (2)$$

We substitute Eq. 1 into Eq. 2 to obtain

*Linear strain rate in y direction*:
$$\varepsilon_{yy} = \frac{d}{dt}\frac{\left(y_{\mathrm{B}} - y_{\mathrm{A}}\right)\left(e^{-bt} - 1\right)}{y_{\mathrm{B}} - y_{\mathrm{A}}} = \frac{d}{dt}\left(e^{-bt} - 1\right) \qquad (3)$$

In the limit as $t \to 0$, we apply the first two terms of the series expansion for $e^{-bt}$,

*Series expansion for* $e^{-bt}$:
$$e^{-bt} = 1 + \left(-bt\right) + \frac{\left(-bt\right)^2}{2!} + \ldots \approx 1 - bt \qquad (4)$$

Finally, for small $t$ we approximate the time derivative as $1/t$, yielding

*Linear strain rate in y direction*:
$$\boxed{\varepsilon_{yy} \to \frac{1}{t}\left(1 - bt - 1\right) = -b} \qquad (5)$$

Comparing to the equation for $\varepsilon$,

*Linear strain rate in y direction*:
$$\boxed{\varepsilon_{yy} = \frac{\partial v}{\partial y} = -b} \qquad (6)$$

Equations 5 and 6 agree, verifying our algebra.

***Discussion*** Since $v$ does not depend on $x$ location in this particular problem, the algebra is simple. In a more general case, both $u$ and $v$ depend on both $x$ and $y$, and a numerical integration scheme is required. We could also have taken the analytical time derivative of Eq. 3, yielding $\varepsilon_{yy} = -be^{-bt}$. Then, as $t \to 0$, $\varepsilon_{xx} \to -b$.

## 4-49E

**Solution** For a given velocity field and an initially square fluid particle, we are to calculate and plot its location and shape after a given time period.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x-y* plane.

***Analysis*** Using the results of Problems 4-43 and 4-46, we can calculate the location of any point on the fluid particle after the elapsed time. We pick 6 points along each edge of the fluid particle, and plot their *x* and *y* locations at $t = 0$ and at $t = 0.2$ s. For example, the point at the lower left corner of the particle is initially at $x = 0.25$ ft and $y = 0.75$ ft at $t = 0$. At $t = 0.2$ s,

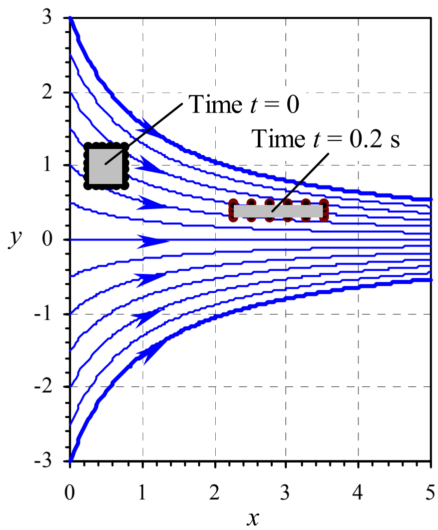


**FIGURE 1**
Movement and distortion of an initially square fluid particle in a converging duct; *x* and *y* are in units of ft. Streamlines (solid blue curves) are also shown for reference.

*x-location of lower left corner of the fluid particle at time t = 0.2 s*:

$$x = \frac{1}{4.6\ 1/\text{s}}\left[\left(5.0\ \text{ft/s} + \left(4.6\ 1/\text{s}\right)\left(0.25\ \text{ft}\right)\right)e^{(4.6\ 1/\text{s})(0.2\ \text{s})} - 5.0\ \text{ft/s}\right] = 2.268\ \text{ft} \qquad (1)$$

and

*y-location of lower left corner of the fluid particle at time t = 0.2 s*:

$$y = \left(0.75\ \text{ft}\right)e^{-(4.6\ 1/\text{s})(0.2\ \text{s})} = 0.2989\ \text{ft} \qquad (2)$$

We repeat the above calculations at all the points along the edges of the fluid particle, and plot both their initial and final positions in Fig. 1 as dots. Finally, we connect the dots to draw the fluid particle shape.

It is clear from the results that the fluid particle shrinks in the *y* direction and stretches in the *x* direction. However, it does not shear or rotate.

***Discussion*** The flow is irrotational since fluid particles do not rotate.

## 4-50E

**Solution** By analyzing the shape of a fluid particle, we are to verify that the given flow field is incompressible.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** Since the flow is two-dimensional, we assume unit depth (1 ft) in the *z* direction (into the page of Fig. P4-49). In the previous problem, we calculated the initial and final locations of several points on the perimeter of an initially square fluid particle. At $t = 0$, the particle volume is

*Fluid particle volume at t = 0 s*:
$$V = (0.50\text{ ft})(0.50\text{ ft})(1.0\text{ ft}) = 0.25\text{ ft}^3 \quad (1)$$

At $t = 0.2$ s, the lower left corner of the fluid particle has moved to $x = 2.2679$ ft, $y = 0.29889$ ft, and the upper right corner has moved to $x = 3.5225$ ft, $y = 0.49815$ ft. Since the fluid particle remains rectangular, we can calculate the fluid particle volume from these two corner locations,

*Fluid particle volume at t = 0.2 s*:
$$V = (3.5225\text{ ft} - 2.2679\text{ ft})(0.49815\text{ ft} - 0.29889\text{ ft})(1.0\text{ ft}) = 0.2500\text{ ft}^3 \quad (2)$$

Thus, to at least four significant digits, the fluid particle volume has not changed, and **the flow is therefore incompressible**.

***Discussion*** The fluid particle stretches in the horizontal direction and shrinks in the vertical direction, but the net volume of the fluid particle does not change.

## 4-51

**Solution** For a given velocity field we are to use volumetric strain rate to verify that the flow field is incompressible..

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The velocity field is

*Velocity field*:
$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \quad (1)$$

We use the equation for volumetric strain rate in Cartesian coordinates, and apply Eq. 1,

*Volumetric strain rate*:
$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = b + (-b) + 0 = 0 \quad (2)$$

Where $\varepsilon_{zz} = 0$ since the flow is two-dimensional. Since the volumetric strain rate is zero everywhere, **the flow is incompressible**.

***Discussion*** The fluid particle stretches in the horizontal direction and shrinks in the vertical direction, but the net volume of the fluid particle does not change.

## 4-52

**Solution** For a given steady two-dimensional velocity field, we are to calculate the $x$ and $y$ components of the acceleration field.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** The velocity field is

$$\vec{V} = (u, v) = (U + a_1 x + b_1 y)\vec{i} + (V + a_2 x + b_2 y)\vec{j} \qquad (1)$$

The acceleration field is obtained from its definition (the material acceleration). The $x$-component is

*x-component of material acceleration*:

$$a_x = \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{Steady}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \underbrace{\cancel{w\frac{\partial u}{\partial z}}}_{\text{Two-D}} = (U + a_1 x + b_1 y)a_1 + (V + a_2 x + b_2 y)b_1 \qquad (2)$$

The $y$-component is

*y-component of material acceleration*:

$$a_y = \underbrace{\cancel{\frac{\partial v}{\partial t}}}_{\text{Steady}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \underbrace{\cancel{w\frac{\partial v}{\partial z}}}_{\text{Two-D}} = (U + a_1 x + b_1 y)a_2 + (V + a_2 x + b_2 y)b_2 \qquad (3)$$

***Discussion*** If there were a $z$-component, it would be treated in the same fashion.

## 4-53

**Solution** We are to find a relationship among the coefficients that causes the flow field to be incompressible.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** We use the equation for volumetric strain rate in Cartesian coordinates, and apply Eq. 1 of Problem 4-52,

*Volumetric strain rate*:

$$\frac{1}{\cancel{V}}\frac{D\cancel{V}}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{\text{Two-D}} = a_1 + b_2 \qquad (1)$$

We recognize that when the volumetric strain rate is zero everywhere, the flow is incompressible. Thus, the desired relationship is

*Relationship to ensure incompressibility*:

$$\boxed{a_1 + b_2 = 0} \qquad (2)$$

***Discussion*** If Eq. 2 is satisfied, the flow is incompressible, regardless of the values of the other coefficients.

**4-54**

**Solution** For a given velocity field we are to calculate the linear strain rates in the $x$ and $y$ directions.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** We use the equations for linear strain rates in Cartesian coordinates, and apply Eq. 1 of Problem 4-52,

*Linear strain rates*:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_1 \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = b_2 \tag{1}$$

***Discussion*** In general, since coefficients $a_1$ and $b_2$ are non-zero, fluid particles stretch (or shrink) in the $x$ and $y$ directions.

**4-55**

**Solution** For a given velocity field we are to calculate the shear strain rate in the $x$-$y$ plane.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** We use the equation for shear strain rate $\varepsilon_{xy}$ in Cartesian coordinates, and apply Eq. 1 of Problem 4-52,

*Shear strain rate in x-y plane*:

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(b_1 + a_2\right) \tag{1}$$

Note that by symmetry $\varepsilon_{yx} = \varepsilon_{xy}$.

***Discussion*** In general, since coefficients $b_1$ and $a_2$ are non-zero, fluid particles distort via shear strain in the $x$ and $y$ directions.

## 4-56

**Solution** For a given velocity field we are to form the 2-D strain rate tensor and determine the conditions necessary for the *x* and *y* axes to be principal axes.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The two-dimensional form of the strain rate tensor is

*2-D strain rate tensor*:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} \qquad (1)$$

We use the linear strain rates and the shear strain rate from the previous two problems to generate the tensor,

*2-D strain rate tensor*:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} = \begin{pmatrix} a_1 & \frac{1}{2}(b_1 + a_2) \\ \frac{1}{2}(b_1 + a_2) & b_2 \end{pmatrix} \qquad (2)$$

If the *x* and *y* axes were principal axes, the diagonals of $\varepsilon_{ij}$ would be non-zero, and the off-diagonals would be zero. Here the off-diagonals go to zero when

*Condition for x and y axes to be principal axes*:

$$b_1 + a_2 = 0 \qquad (3)$$

***Discussion*** For the more general case in which Eq. 3 is not satisfied, the principal axes can be calculated using tensor algebra.

## 4-57

**Solution** For a given velocity field we are to calculate the vorticity vector and discuss its orientation.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** We use the equation for vorticity vector $\vec{\zeta}$ in Cartesian coordinates, and apply Eq. 1 of Problem 4-52,

*Vorticity vector*:

$$\vec{\zeta} = \left( \underbrace{\cancel{\frac{\partial w}{\partial y}}}_{\text{Two-D}} - \underbrace{\cancel{\frac{\partial v}{\partial z}}}_{\text{Two-D}} \right)\vec{i} + \left( \underbrace{\cancel{\frac{\partial u}{\partial z}}}_{\text{Two-D}} - \underbrace{\cancel{\frac{\partial w}{\partial x}}}_{\text{Two-D}} \right)\vec{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\vec{k} = (a_2 - b_1)\vec{k} \qquad (1)$$

The only non-zero component of vorticity is in the **z (or –z) direction**.

***Discussion*** For any two-dimensional flow in the *x*-*y* plane, the vorticity vector *must* point in the *z* (or –*z*) direction. The sign of the *z*-component of vorticity in Eq. 1 obviously depends on the sign of $a_2 - b_1$.

## 4-58

**Solution** For the given velocity field we are to calculate the two-dimensional linear strain rates from fundamental principles and compare with the given equation.

***Assumptions*** **1** The flow is incompressible. **2** The flow is steady. **3** The flow is two-dimensional.

***Analysis*** First, for convenience, we number the equations in the problem statement:

*Velocity field*: $$\vec{V} = (u,v) = (a+by)\vec{i} + 0\vec{j} \qquad (1)$$

*Lower left corner at t + dt*: $$\left(x + (a+by)dt,\, y\right) \qquad (2)$$

*Linear strain rate in Cartesian coordinates*: $$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad (3)$$

(*a*) The lower right corner of the fluid particle moves the same amount as the lower left corner since $u$ does not depend on $y$ position. Thus,

*Lower right corner at t + dt*: $$\left(x + dx + (a+by)dt,\, y\right) \qquad (4)$$

Similarly, the top two corners of the fluid particle move to the right at speed $a + b(y+dy)dt$. Thus,

*Upper left corner at t + dt*: $$\left(x + \left(a + b(y+dy)\right)dt,\, y+dy\right) \qquad (5)$$

and

*Upper right corner at t + dt*: $$\left(x + dx + \left(a + b(y+dy)\right)dt,\, y+dy\right) \qquad (6)$$

(*b*) From the fundamental definition of linear strain rate in the $x$-direction, we consider the lower edge of the fluid particle. Its rate of increase in length divided by its original length is found by using Eqs. 2 and 4,

$\varepsilon_{xx}$: $$\varepsilon_{xx} = \frac{1}{dt}\left[\frac{\overbrace{x+dx+(a+by)dt-\left(x+(a+by)dt\right)}^{\text{Length of lower edge at } t+dt} - \overbrace{dx}^{\text{Length of lower edge at } t}}{dx}\right] = 0 \qquad (6)$$

We get the same result by considering the *upper* edge of the fluid particle. Similarly, using the left edge of the fluid particle and Eqs. 2 and 5 we get

$\varepsilon_{yy}$: $$\varepsilon_{yy} = \frac{1}{dt}\left[\frac{\overbrace{y+dy-y}^{\text{Length of left edge at } t+dt} - \overbrace{dy}^{\text{Length of left edge at } t}}{dy}\right] = 0 \qquad (7)$$

We get the same result by considering the *right* edge of the fluid particle. Thus both the $x$- and $y$-components of linear strain rate are zero for this flow field.

(*c*) From Eq. 3 we calculate

*Linear strain rates*: $$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \tag{8}$$

***Discussion*** Although the algebra in this problem is rather straight-forward, it is good practice for the more general case (a later problem).

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## 4-59

**Solution** We are to verify that the given flow field is incompressible using two different methods.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional.

***Analysis***

(*a*) The volume of the fluid particle at time *t* is.

*Volume at time t*: $$\mathcal{V}(t) = dxdydz \tag{1}$$

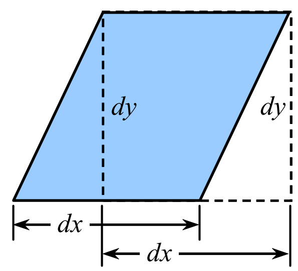


**FIGURE 1**
The area of a rhombus is equal to its base times its height, which here is *dxdy*.

where *dz* is the length of the fluid particle in the *z* direction. At time $t + dt$, we assume that the fluid particle's dimension *dz* remains fixed since the flow is two-dimensional. Thus its volume is *dz* times the area of the rhombus shown in Fig. P4-58, as illustrated in Fig. 1,

*Volume at time t + dt*: $$\mathcal{V}(t+dt) = dxdydz \tag{2}$$

Since Eqs. 1 and 2 are equal, the volume of the fluid particle has not changed, and **the flow is therefore incompressible**.

(*b*) We use the equation for volumetric strain rate in Cartesian coordinates, and apply the results of the previous problem,

*Volumetric strain rate*: $$\frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0 + 0 + 0 = 0 \tag{3}$$

Where $\varepsilon_{zz} = 0$ since the flow is two-dimensional. Since the volumetric strain rate is zero everywhere, **the flow is incompressible**.

***Discussion*** Although the fluid particle deforms with time, its height, its depth, and the length of its horizontal edges remain constant.

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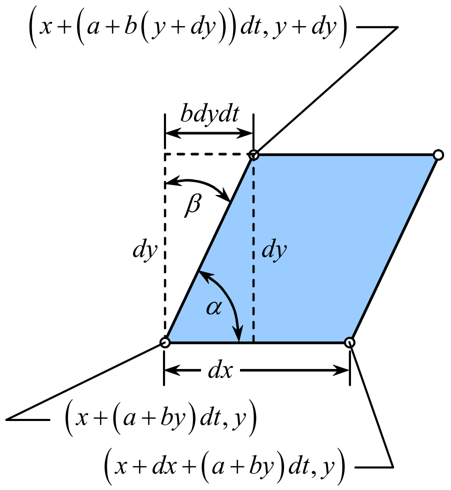


**FIGURE 1**
A magnified view of the deformed fluid particle at time $t + dt$, with the location of three corners indicated, and angles $\alpha$ and $\beta$ defined.

## 4-60

**Solution** For the given velocity field we are to calculate the two-dimensional shear strain rate in the *x-y* plane from fundamental principles and compare with the given equation.

***Assumptions*** **1** The flow is incompressible. **2** The flow is steady. **3** The flow is two-dimensional.

***Analysis***
(*a*) The shear strain rate is

*Shear strain rate in Cartesian coordinates*:
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (1)$$

From the fundamental definition of shear strain rate in the *x-y* plane, we consider the bottom edge and the left edge of the fluid particle, which intersect at 90° at the lower left corner at time *t* (Fig. P4-58). We define angle $\alpha$ between the lower edge and the left edge of the fluid particle, and angle $\beta$, the complement of $\alpha$ (Fig. 1). The rate of decrease of angle $\alpha$ over time interval *dt* is obtained from application of trigonometry. First, we calculate angle $\beta$,

*Angle $\beta$ at time t + dt*:
$$\beta = \arctan\left(\frac{bdydt}{dy}\right) = \arctan\left(bdt\right) \approx bdt \qquad (2)$$

The approximation is valid for very small angles. As the time interval $dt \to 0$, Eq. 2 is correct. At time $t + dt$, angle $\alpha$ is

*Angle $\alpha$ at time t + dt*:
$$\alpha = \frac{\pi}{2} - \beta \approx \frac{\pi}{2} - bdt \qquad (3)$$

During this time interval, $\alpha$ changes from 90° ($\pi$/2 radians) to the expression given by Eq. 2. Thus the rate of change of $\alpha$ is

*Rate of change of angle $\alpha$*:
$$\frac{d\alpha}{dt} = \frac{1}{dt}\left[\underbrace{\left(\frac{\pi}{2} - bdt\right)}_{\alpha \text{ at } t+dt} - \underbrace{\frac{\pi}{2}}_{\alpha \text{ at } t}\right] = -b \qquad (4)$$

Finally, since shear strain rate is defined as *half* of the rate of *decrease* of angle $\alpha$,

*Shear strain rate*:
$$\varepsilon_{xy} = -\frac{1}{2}\frac{d\alpha}{dt} = \frac{b}{2} \qquad (5)$$

(*b*) From Eq. 1 we calculate

*Shear strain rate*:
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(b + 0\right) = \frac{b}{2} \qquad (6)$$

Both methods for obtaining the shear strain rate agree (Eq. 5 and Eq. 6).

***Discussion*** Although the algebra in this problem is rather straight-forward, it is good practice for the more general case (a later problem).

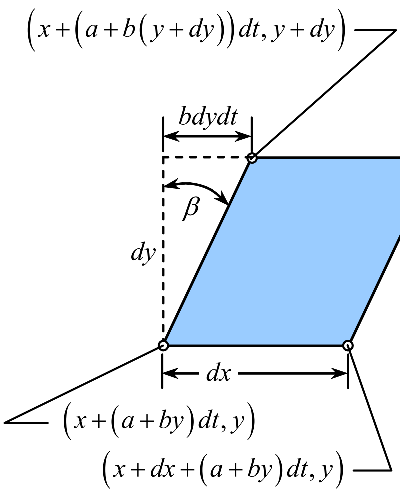


**FIGURE 1**
A magnified view of the deformed fluid particle at time $t + dt$, with the location of three corners indicated, and angle $\beta$ defined.

## 4-61

**Solution** For the given velocity field we are to calculate the two-dimensional rate of rotation in the *x*-*y* plane from fundamental principles and compare with the given equation.

***Assumptions*** **1** The flow is incompressible. **2** The flow is steady. **3** The flow is two-dimensional.

***Analysis***
(*a*) The rate of rotation in Cartesian coordinates is

*Rate of rotation in Cartesian coordinates*:
$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1}$$

From the fundamental definition of rate of rotation in the *x*-*y* plane, we consider the bottom edge and the left edge of the fluid particle, which intersect at 90° at the lower left corner at time *t* (Fig. P4-58). We define angle $\beta$ in Fig. 1, where $\beta$ is the negative of the angle of rotation of the left edge of the fluid particle (negative because rotation is mathematically positive in the *counterclockwise* direction). We calculate angle $\beta$ using trigonometry,

*Angle $\beta$ at time t + dt*:
$$\beta = \arctan\left(\frac{bdydt}{dy}\right) = \arctan\left(bdt\right) \approx bdt \tag{2}$$

The approximation is valid for very small angles. As the time interval $dt \to 0$, Eq. 2 is correct.

Meanwhile, the bottom edge of the fluid particle has not rotated at all. Thus, the average angle of rotation of the two line segments (lower and left edges) at time $t + dt$ is

*Average angle of rotation at time t + dt*:
$$AVG = \frac{1}{2}\left(0 - \beta\right) \approx -\frac{b}{2}dt \tag{3}$$

Thus the average rotation *rate* during time interval *dt* is

*Rate of rotation in x-y plane*:
$$\omega_z = \frac{d\left(AVG\right)}{dt} = \frac{1}{dt}\left(-\frac{b}{2}dt\right) = -\frac{b}{2} \tag{4}$$

(*b*) From Eq. 1 we calculate

*Rate of rotation*:
$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(0 - b\right) = -\frac{b}{2} \tag{5}$$

Both methods for obtaining the rate of rotation agree (Eq. 4 and Eq. 5).

***Discussion*** The rotation rate is negative, indicating *clockwise* rotation about the *z*-axis. This agrees with our intuition as we follow the fluid particle in Fig. P4-58.

## 4-62

**Solution** We are to determine whether the shear flow of Problem 4-22 is rotational or irrotational, and we are to calculate the vorticity in the $z$ direction.

***Analysis***

($a$) Since the rate of rotation is non-zero, it means that **the flow is *rotational***.

($b$) Vorticity is defined as twice the rate of rotation, or twice the angular velocity. In the $z$ direction,

*Vorticity component*: $$\zeta_z = 2\omega_z = 2\left(-\frac{b}{2}\right) = -b \qquad (1)$$

***Discussion*** Vorticity is negative, indicating *clockwise* rotation about the $z$-axis.

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## 4-63

**Solution** We are to prove the above expression for flow in the $xy$-plane.

***Assumptions*** **1** The flow is incompressible and two-dimensional.

***Analysis*** For flow in the $xy$-plane, we are to show that:

*Rate of rotation*: $$\omega = \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (1)$$

By definition, the rate of rotation (angular velocity) at a point is the average rotation rate of two initially perpendicular lines that intersect at the point. In this particular problem, Line a (PA) and Line b (PB) are initially perpendicular, and intersect at point P. Line a rotates by angle $\alpha_a$, and Line b rotates by angle $\alpha_b$, Thus, the average angle of rotation is

*Average angle of rotation*: $$\frac{\alpha_a + \alpha_b}{2} \qquad (2)$$

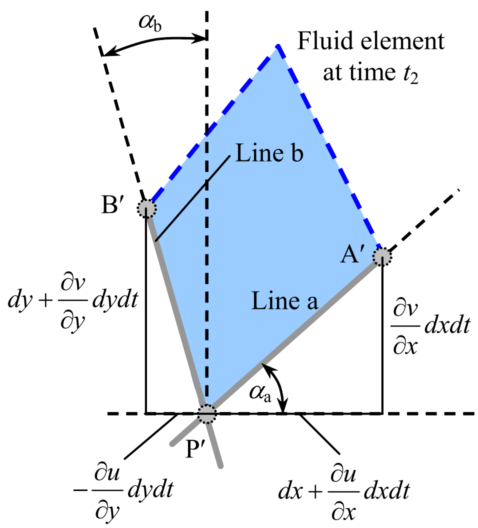


**FIGURE 1**
A close-up view of the distorted fluid element at time $t_2$.

During time increment $dt$, point P moves a distance $udt$ to the right and $vdt$ up (to first order, assuming $dt$ is very small). Similarly, point A moves a distance $\left(u + \frac{\partial u}{\partial x} dx\right) dt$ to the right and $\left(v + \frac{\partial v}{\partial x} dx\right) dt$ up, and point B moves a distance $\left(u + \frac{\partial u}{\partial y} dy\right) dt$ to the right and $\left(v + \frac{\partial v}{\partial y} dy\right) dt$ up. Since point A is initially at distance $dx$ to the right of point P, its position to the right of point P′ at the later time $t_2$ is

*Horizontal distance from point P′ to point A′ at time $t_2$:* $\quad dx + \frac{\partial u}{\partial x} dxdt \quad$ (3)

On the other hand, point A is at the same *vertical* level as point P at time $t_1$. Thus, the vertical distance from point P′ to point A′ at time $t_2$ is

*Vertical distance from point P′ to point A′ at time $t_2$:* $\quad \frac{\partial v}{\partial x} dxdt \quad$ (3)

Similarly, point B is located at distance $dy$ vertically above point P at time $t_1$, and thus we write

*Horizontal distance from point P′ to point B′ at time $t_2$:* $\quad -\frac{\partial u}{\partial y} dydt \quad$ (4)

and

*Vertical distance from point P′ to point B′ at time $t_2$:* $\quad dy + \frac{\partial v}{\partial y} dydt \quad$ (5)

We mark the horizontal and vertical distances between point A′ and point P′ and between point B′ and point P′ at time $t_2$ in Fig. 1. From the figure we see that

*Angle $\alpha_a$ in terms of velocity components*:

$$\alpha_a = \tan^{-1}\left(\frac{\frac{\partial v}{\partial x} dxdt}{dx + \frac{\partial u}{\partial x} dxdt}\right) \approx \tan^{-1}\left(\frac{\frac{\partial v}{\partial x} dxdt}{dx}\right) = \tan^{-1}\left(\frac{\partial v}{\partial x} dt\right) \approx \frac{\partial v}{\partial x} dt \qquad (6)$$

The first approximation in Eq. 6 is due to the fact that as the size of the fluid element shrinks to a point, $dx \to 0$, and at the same time $dt \to 0$. Thus, the second term in the denominator is second-order compared to the first-order term $dx$ and can be neglected. The second approximation in Eq. 6 is because as $dt \to 0$ angle $\alpha_a$ is very small, and $\tan\alpha_a \to \alpha_a$. Similarly,

*Angle $\alpha_b$ in terms of velocity components*:

$$\alpha_b = \tan^{-1}\left(\frac{-\frac{\partial u}{\partial y} dydt}{dy + \frac{\partial v}{\partial y} dydt}\right) \approx \tan^{-1}\left(\frac{-\frac{\partial u}{\partial y} dydt}{dy}\right) = \tan^{-1}\left(-\frac{\partial u}{\partial y} dt\right) \approx -\frac{\partial u}{\partial y} dt \qquad (7)$$

Finally then, the average rotation angle (Eq. 2) becomes

*Average angle of rotation*: $$\frac{\alpha_{\mathrm{a}}+\alpha_{\mathrm{b}}}{2}=\frac{1}{2}\left(\frac{\partial v}{\partial x}dt-\frac{\partial u}{\partial y}dt\right)=\frac{dt}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \qquad (8)$$

and the average *rate* of rotation (angular velocity) of the fluid element about point P in the *x*-*y* plane becomes

$$\omega=\omega_z=\frac{d}{dt}\left(\frac{\alpha_{\mathrm{a}}+\alpha_{\mathrm{b}}}{2}\right)=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \qquad (9)$$

***Discussion*** Eq. 9 can be easily extended to three dimensions by performing a similar analysis in the *x*-*z* plane and in the *y*-*z* plane.

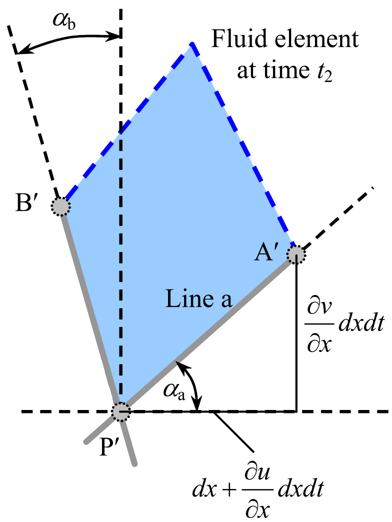


**FIGURE 1**
A close-up view of the distorted fluid element at time $t_2$.

**4-64**

**Solution** We are to prove the above expression.

***Assumptions*** **1** The flow is incompressible and two-dimensional.

***Analysis*** We are to prove the following:

*Linear strain rate in x-direction*: $$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{1}$$

By definition, the rate of linear strain is the rate of increase in length of a line segment in a given direction divided by the original length of the line segment in that direction. During time increment $dt$, point P moves a distance $udt$ to the right and $vdt$ up (to first order, assuming $dt$ is very small). Similarly, point A moves a distance $\left(u + \frac{\partial u}{\partial x}dx\right)dt$ to the right and $\left(v + \frac{\partial v}{\partial x}dx\right)dt$ up. Since point A is initially at distance $dx$ to the right of point P, its position to the right of point P′ at the later time $t_2$ is

*Horizontal distance from point P′ to point A′ at time $t_2$*: $$dx + \frac{\partial u}{\partial x}dxdt \tag{2}$$

On the other hand, point A is at the same *vertical* level as point P at time $t_1$. Thus, the vertical distance from point P′ to point A′ at time $t_2$ is

*Vertical distance from point P′ to point A′ at time $t_2$*: $$\frac{\partial v}{\partial x}dxdt \tag{3}$$

We mark the horizontal and vertical distances between point A′ and point P′ at time $t_2$ in Fig. 1. From the figure we see that

*Linear strain rate in the x direction as line PA changes to P′A′*:

$$\varepsilon_{xx} = \frac{d}{dt}\left(\frac{\overbrace{dx + \frac{\partial u}{\partial x}dxdt}^{\text{Length of P′A′ in } x \text{ direction}} - \overbrace{\mathrm{dx}}^{\text{Length of PA in } x \text{ direction}}}{\underbrace{dx}_{\text{Length of PA in } x \text{ direction}}}\right) = \frac{d}{dt}\left(\frac{\partial u}{\partial x}dt\right) = \frac{\partial u}{\partial x} \tag{4}$$

Thus **Eq. 1 is verified**.

***Discussion*** The distortion of the fluid element is exaggerated in Fig. 1. As time increment $dt$ and fluid element length $dx$ approach zero, the first-order approximations become exact.

**4-65**

**Solution** We are to prove the above expression.

***Assumptions*** **1** The flow is incompressible and two-dimensional.

***Analysis*** We are to prove the following:

*Shear strain rate in xy-plane*:
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (1)$$

By definition, the shear strain rate at a point is half of the rate of decrease of the angle between two initially perpendicular lines that intersect at the point. In Fig. P4-63, Line a (PA) and Line b (PB) are initially perpendicular, and intersect at point P. Line a rotates by angle $\alpha_a$, and Line b rotates by angle $\alpha_b$. The angle between these two lines changes from $\pi/2$ at time $t_1$ to $\alpha_{a\text{-}b}$ at time $t_2$ as sketched in Fig. 1. The shear strain rate at point P for initially perpendicular lines in the $x$ and $y$ directions is thus

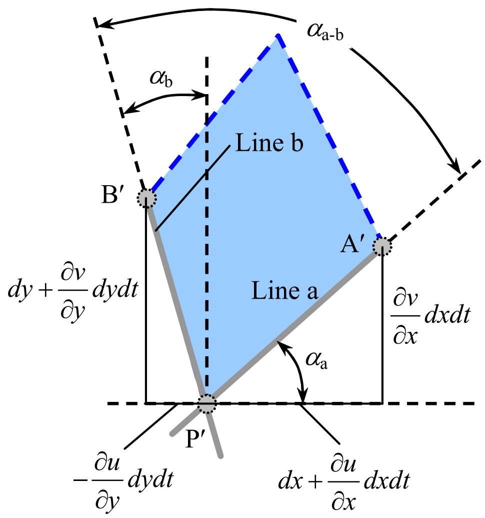


**FIGURE 1**
A close-up view of the distorted fluid element at time $t_2$.

*Shear strain rate, initially perpendicular lines in the x and y directions*:
$$\varepsilon_{xy} = -\frac{1}{2}\frac{d}{dt}\alpha_{a\text{-}b} \qquad (2)$$

During time increment $dt$, point P moves a distance $udt$ to the right and $vdt$ up (to first order, assuming $dt$ is very small). Similarly, point A moves a distance $\left(u + \frac{\partial u}{\partial x}dx\right)dt$ to the right and $\left(v + \frac{\partial v}{\partial x}dx\right)dt$ up, and point B moves a distance $\left(u + \frac{\partial u}{\partial y}dy\right)dt$ to the right and $\left(v + \frac{\partial v}{\partial y}dy\right)dt$ up. Since point A is initially at distance $dx$ to the right of point P, its position to the right of point P′ at the later time $t_2$ is

*Horizontal distance from point P′ to point A′ at time $t_2$*:
$$dx + \frac{\partial u}{\partial x}dxdt \qquad (3)$$

On the other hand, point A is at the same *vertical* level as point P at time $t_1$. Thus, the vertical distance from point P′ to point A′ at time $t_2$ is

*Vertical distance from point P′ to point A′ at time $t_2$*:
$$\frac{\partial v}{\partial x}dxdt \qquad (3)$$

Similarly, point B is located at distance $dy$ vertically above point P at time $t_1$, and thus we write

*Horizontal distance from point P′ to point B′ at time $t_2$*:
$$-\frac{\partial u}{\partial y}dydt \qquad (4)$$

and

*Vertical distance from point P′ to point B′ at time $t_2$*:
$$dy + \frac{\partial v}{\partial y}dydt \qquad (5)$$

We mark the horizontal and vertical distances between point A′ and point P′ and between point B′ and point P′ at time $t_2$ in Fig. 1. From the figure we see that

*Angle $\alpha_a$ in terms of velocity components*:

$$\alpha_{\text{a}} = \tan^{-1}\left(\frac{\frac{\partial v}{\partial x}dxdt}{dx+\frac{\partial u}{\partial x}dxdt}\right) \approx \tan^{-1}\left(\frac{\frac{\partial v}{\partial x}dxdt}{dx}\right) = \tan^{-1}\left(\frac{\partial v}{\partial x}dt\right) \approx \frac{\partial v}{\partial x}dt \qquad (6)$$

The first approximation in Eq. 6 is due to the fact that as the size of the fluid element shrinks to a point, $dx \to 0$, and at the same time $dt \to 0$. Thus, the second term in the denominator is second-order compared to the first-order term $dx$ and can be neglected. The second approximation in Eq. 6 is because as $dt \to 0$ angle $\alpha_a$ is very small, and $\tan\alpha_a \to \alpha_a$. Similarly,

*Angle $\alpha_b$ in terms of velocity components*:

$$\alpha_{\text{b}} = \tan^{-1}\left(\frac{-\frac{\partial u}{\partial y}dydt}{dy+\frac{\partial v}{\partial y}dydt}\right) \approx \tan^{-1}\left(\frac{-\frac{\partial u}{\partial y}dydt}{dy}\right) = \tan^{-1}\left(-\frac{\partial u}{\partial y}dt\right) \approx -\frac{\partial u}{\partial y}dt \qquad (7)$$

Angle $\alpha_{\text{a-b}}$ at time $t_2$ is calculated from Fig. 1 as

*Angle $\alpha_{\text{a-b}}$ at time $t_2$ in terms of velocity components*:

$$\alpha_{\text{a-b}} = \frac{\pi}{2} + \alpha_{\text{b}} - \alpha_{\text{a}} = \frac{\pi}{2} - \frac{\partial u}{\partial y}dt - \frac{\partial v}{\partial x}dt \qquad (8)$$

where we have used Eqs. 6 and 7. Finally then, the shear strain rate (Eq. 2) becomes

*Shear strain rate, initially perpendicular lines in the x and y directions*:

$$\varepsilon_{xy} = -\frac{1}{2}\frac{d}{dt}\alpha_{\text{a-b}} \approx -\frac{1}{2}\frac{1}{dt}\left(\overbrace{\frac{\pi}{2} - \frac{\partial u}{\partial y}dt - \frac{\partial v}{\partial x}dt}^{\alpha_{\text{a-b}}\text{ at }t_2} - \overbrace{\frac{\pi}{2}}^{\alpha_{\text{a-b}}\text{ at }t_1}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (9)$$

which agrees with Eq. 1. Thus, **Eq. 1 is proven**.

***Discussion*** Eq. 9 can be easily extended to three dimensions by performing a similar analysis in the *x-z* plane and in the *y-z* plane.

**4-66**

**Solution** For a given linear strain rate in the *x*-direction, we are to calculate the linear strain rate in the *y*-direction.

***Analysis*** Since the flow is incompressible, the volumetric strain rate must be zero. In two dimensions,

*Volumetric strain rate in the x-y plane*:
$$\frac{1}{V}\frac{DV}{Dt}=\varepsilon_{xx}+\varepsilon_{yy}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

Thus, the linear strain rate in the *y*-direction is the negative of that in the *x*-direction,

*Linear strain rate in y-direction*:
$$\varepsilon_{yy}=\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}=\mathbf{-2.5\ 1/s} \qquad (2)$$

***Discussion*** The fluid element *stretches* in the *x*-direction since $\varepsilon_{xx}$ is positive. Because the flow is incompressible, the fluid element must *shrink* in the *y*-direction, yielding a value of $\varepsilon_{yy}$ that is negative.

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**4-67**

**Solution** We are to calculate the vorticity of fluid particles in a tank rotating in solid body rotation about its vertical axis.

***Assumptions*** **1** The flow is steady. **2** The *z*-axis is in the vertical direction.

***Analysis*** Vorticity $\vec{\zeta}$ is twice the angular velocity $\vec{\omega}$. Here,

*Angular velocity*:
$$\vec{\omega}=360\frac{\text{rot}}{\text{min}}\left(\frac{1\text{ min}}{60\text{ s}}\right)\left(\frac{2\pi\text{ rad}}{\text{rot}}\right)\vec{k}=37.70\vec{k}\ \text{rad/s} \qquad (1)$$

where $\vec{k}$ is the unit vector in the vertical (*z*) direction. The vorticity is thus

*Vorticity*:
$$\vec{\zeta}=2\vec{\omega}=2\times 37.70\vec{k}\ \text{rad/s}=75.4\vec{k}\ \text{rad/s} \qquad (2)$$

***Discussion*** Because the water rotates as a solid body, the vorticity is constant throughout the tank, and points vertically upward.

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## 4-68

**Solution** We are to calculate the angular speed of a tank rotating about its vertical axis.

***Assumptions*** **1** The flow is steady. **2** The *z*-axis is in the vertical direction.

***Analysis*** Vorticity $\vec{\zeta}$ is twice the angular velocity $\vec{\omega}$. Thus,

*Angular velocity*:
$$\vec{\omega} = \frac{\vec{\zeta}}{2} = \frac{-55.4\vec{k}\ \text{rad/s}}{2} = -27.7\vec{k}\ \text{rad/s} \qquad (1)$$

where $\vec{k}$ is the unit vector in the vertical ($z$) direction. The angular velocity is negative, which by definition is in the clockwise direction about the vertical axis. We express the rate of rotation in units of rpm,

*Rate of rotation*:
$$\dot{n} = -27.7\frac{\text{rad}}{\text{s}}\left(\frac{60\ \text{s}}{1\ \text{min}}\right)\left(\frac{\text{rot}}{2\pi\ \text{rad}}\right) = -265\frac{\text{rot}}{\text{min}} = \mathbf{-265\ rpm} \qquad (2)$$

***Discussion*** Because the vorticity is constant throughout the tank, the water rotates as a solid body.

## 4-69

**Solution** For a tank of given rim radius and speed, we are to calculate the magnitude of the component of vorticity in the vertical direction.

***Assumptions*** **1** The flow is steady. **2** The *z*-axis is in the vertical direction.

***Analysis*** The linear speed at the rim is equal to $r_{\text{rim}}\omega_z$. Thus,

*Component of angular velocity in z-direction*:
$$\omega_z = \frac{V_{\text{rim}}}{r_{\text{rim}}} = \frac{2.6\ \text{m/s}}{0.35\ \text{m}} = 7.429\ \text{rad/s} \qquad (1)$$

Vorticity $\vec{\zeta}$ is twice the angular velocity $\vec{\omega}$. Thus,

*z-component of vorticity*:
$$\zeta_z = 2\omega_z = 2(7.429\ \text{rad/s}) = 14.86\ \text{rad/s} \cong \mathbf{15.0\ rad/s} \qquad (2)$$

***Discussion*** Radian is a non-dimensional unit, so we can insert it into Eq. 1. The final answer is given to two significant digits for consistency with the given information.

## 4-70C

**Solution** We are to explain the relationship between vorticity and rotationality.

***Analysis*** Vorticity is a measure of the rotationality of a fluid particle. If a particle rotates, its vorticity is non-zero. Mathematically, **the vorticity vector is twice the angular velocity vector**.

***Discussion*** If the vorticity is zero, the flow is called irrotational.

## 4-71

**Solution** For a given deformation of a fluid particle in one direction, we are to calculate its deformation in the other direction.

***Assumptions*** **1** The flow is incompressible. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** Since the flow is incompressible and two-dimensional, the area of the fluid element must remain constant (volumetric strain rate must be zero in an incompressible flow). The area of the original fluid particle is $a^2$. Hence, **the vertical dimension of the fluid particle at the later time must be $a^2/2a = a/2$**.

***Discussion*** Since the particle stretches by a factor of two in the *x*-direction, it shrinks by a factor of two in the *y*-direction.

## 4-72

**Solution** We are to calculate the percentage change in fluid density for a fluid particle undergoing two-dimensional deformation.

***Assumptions*** **1** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The area of the original fluid particle is $a^2$. Assuming that the mass of the fluid particle is *m* and its dimension in the *z*-direction is also *a*, the initial density is $\rho = m/V = m/a^3$. As the particle moves and deforms, its mass must remain constant. If its dimension in the *z*-direction remains equal to *a*, the density at the later time is

*Density at the later time*:
$$\rho = \frac{m}{V} = \frac{m}{(1.06a)(0.931a)(a)} = 1.013\frac{m}{a^3} \tag{1}$$

Compared to the original density, **the density has increased by about 1.3%**.

***Discussion*** The fluid particle has stretched in the *x*-direction and shrunk in the *y*-direction, but there is nevertheless a net decrease in volume, corresponding to a net increase in density.

## 4-73

**Solution** For a given velocity field we are to calculate the vorticity.

***Analysis*** The velocity field is

$$\vec{V} = (u, v, w) = (3.0 + 2.0x - y)\vec{i} + (2.0x - 2.0y)\vec{j} + (0.5xy)\vec{k} \tag{1}$$

In Cartesian coordinates, the vorticity vector is

*Vorticity vector in Cartesian coordinates*:

$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} \tag{2}$$

We substitute the velocity components $u = 3.0 + 2.0x - y$, $v = 2.0x - 2.0y$, and $w = 0.5xy$ from Eq. 1 into Eq. 2 to obtain

*Vorticity vector*:

$$\begin{aligned}\vec{\zeta} &= (0.5x - 0)\vec{i} + (0 - 0.5y)\vec{j} + (2.0 - (-1))\vec{k} \\ &= (0.5x)\vec{i} - (0.5y)\vec{j} + (3.0)\vec{k}\end{aligned} \tag{3}$$

***Discussion*** The vorticity is non-zero implying that this flow field is *rotational*.

## 4-74

**Solution** We are to determine if the flow is rotational, and if so calculate the *z*-component of vorticity.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* plane.

***Analysis*** The velocity field is given by

*Velocity field, Couette flow*:

$$\vec{V} = (u, v) = \left(V\frac{y}{h}\right)\vec{i} + 0\vec{j} \tag{1}$$

If the vorticity is non-zero, the flow is rotational. So, we calculate the *z*-component of vorticity,

*z-component of vorticity*:

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - \frac{V}{h} = -\frac{V}{h} \tag{2}$$

Since vorticity is non-zero, **this flow is rotational**. Furthermore, the vorticity is negative, implying that **particles rotate in the clockwise direction**.

***Discussion*** The vorticity is constant at every location in this flow.

## 4-75

**Solution** For the given velocity field for Couette flow, we are to calculate the two-dimensional linear strain rates and the shear strain rate.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The linear strain rates in the *x* direction and in the *y* direction are

*Linear strain rates*: $$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \mathbf{0} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \mathbf{0} \qquad (1)$$

The shear strain rate in the *x*-*y* plane is

*Shear strain rate*: $$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{V}{h} + 0\right) = \frac{V}{2h} \qquad (2)$$

Fluid particles in this flow have non-zero shear strain rate.

***Discussion*** Since the linear strain rates are zero, fluid particles deform (shear), but do not *stretch* in either the horizontal or vertical directions.

## 4-76

**Solution** For the Couette flow velocity field we are to form the 2-D strain rate tensor and determine if the *x* and *y* axes are principal axes.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The two-dimensional strain rate tensor, $\varepsilon_{ij}$, is

*2-D strain rate tensor*: $$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} \qquad (1)$$

We use the linear strain rates and the shear strain rate from the previous problem to generate the tensor,

*2-D strain rate tensor*: $$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{V}{2h} \\ \dfrac{V}{2h} & 0 \end{pmatrix} \qquad (2)$$

Note that by symmetry $\varepsilon_{yx} = \varepsilon_{xy}$. If the *x* and *y* axes were principal axes, the diagonals of $\varepsilon_{ij}$ would be non-zero, and the off-diagonals would be zero. Here we have the opposite case, so **the x and y axes are not principal axes**.

***Discussion*** The principal axes can be calculated using tensor algebra.

## Reynolds Transport Theorem

### 4-77C

**Solution**

(*a*) **False**: The statement is backwards, since the conservation laws are naturally occurring in the system form.

(*b*) **False**: The RTT can be applied to any control volume, fixed, moving, or deforming.

(*c*) **True**: The RTT has an unsteady term and can be applied to unsteady problems.

(d) **True**: The extensive property $B$ (or its intensive form $b$) in the RTT can be any property of the fluid – scalar, vector, or even tensor.

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### 4-78

**Solution** For the case in which $B_{sys}$ is the mass $m$ of a system, we are to use the RTT to derive the equation of conservation of mass for a control volume.

***Analysis*** The general form of the Reynolds transport theorem is given by

*General form of the RTT*:
$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b d\mathcal{V} + \int_{CS} \rho b \vec{V}_r \cdot \vec{n} dA \qquad (1)$$

Setting $B_{sys} = m$ means that $b = m/m = 1$. Plugging these and $dm/dt = 0$ into Eq. 1 yields

*Conservation of mass for a CV*:
$$\boxed{0 = \frac{d}{dt}\int_{CV} \rho d\mathcal{V} + \int_{CS} \rho \vec{V}_r \cdot \vec{n} dA} \qquad (2)$$

***Discussion*** Eq. 2 is general and applies to any control volume – fixed, moving, or even deforming.

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## 4-79

**Solution** For the case in which $B_{sys}$ is the linear momentum $m\vec{V}$ of a system, we are to use the RTT to derive the equation of conservation of linear momentum for a control volume.

***Analysis*** Newton's second law is

*Newton's second law for a system*:
$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(m\vec{V}\right)_{sys} \tag{1}$$

Setting $B_{sys} = m\vec{V}$ means that $b = m\vec{V}/m = \vec{V}$. Plugging these and Eq. 1 into the equation of Problem 4-78 yields

$$\sum \vec{F} = \frac{d}{dt}\left(m\vec{V}\right)_{sys} = \frac{d}{dt}\int_{CV} \rho\vec{V}d\mathcal{V} + \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right)dA$$

or simply

*Conservation of linear momentum for a CV*:
$$\boxed{\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V}d\mathcal{V} + \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right)dA} \tag{2}$$

***Discussion*** Eq. 2 is general and applies to any control volume – fixed, moving, or even deforming.

## 4-80

**Solution** For the case in which $B_{sys}$ is the angular momentum $\vec{H}$ of a system, we are to use the RTT to derive the equation of conservation of angular momentum for a control volume.

***Analysis*** The conservation of angular momentum is expressed as

*Conservation of angular momentum for a system*: $$\sum \vec{M} = \frac{d}{dt}\vec{H}_{sys} \quad (1)$$

Setting $B_{sys} = \vec{H}$ means that $b = \left(\vec{r} \times m\vec{V}\right)/m = \vec{r} \times \vec{V}$, noting that $m$ = constant for a system. Plugging these and Eq. 1 into the equation of Problem 4-78 yields

$$\sum \vec{M} = \frac{d}{dt}\vec{H}_{sys} = \frac{d}{dt}\int_{CV} \rho\left(\vec{r} \times \vec{V}\right) dV + \int_{CS} \rho\left(\vec{r} \times \vec{V}\right)\left(\vec{V}_r \cdot \vec{n}\right) dA$$

or simply

*Conservation of angular momentum for a CV*:

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} \rho\left(\vec{r} \times \vec{V}\right) dV + \int_{CS} \rho\left(\vec{r} \times \vec{V}\right)\left(\vec{V}_r \cdot \vec{n}\right) dA \quad (2)$$

***Discussion*** Eq. 2 is general and applies to any control volume – fixed, moving, or even deforming.

**4-81**

**Solution** $F(t)$ is to be evaluated from the given expression.

***Analysis*** The integral is

$$F(t) = \frac{d}{dt}\int_{x=At}^{x=Bt} e^{-2x^2}\,dx \qquad (1)$$

We could try integrating first, and then differentiating, but we can instead use the 1-D Leibnitz theorem. Here, $G(x,t) = e^{-2x^2}$ ($G$ is not a function of time in this simple example). The limits of integration are $a(t) = At$ and $b(t) = Bt$. Thus,

$$\begin{aligned} F(t) &= \int_a^b \frac{\partial G}{\partial t}dx + \frac{db}{dt}G(b,t) - \frac{da}{dt}G(a,t) \\ &= \quad 0 \quad + Be^{-2b^2} \quad - Ae^{-2a^2} \end{aligned} \qquad (2)$$

or

$$\boxed{F(t) = Be^{-B^2t^2} - Ae^{-A^2t^2}} \qquad (3)$$

***Discussion*** You are welcome to try to obtain the same solution without using the Leibnitz theorem.

## Review Problems

### 4-82

**Solution** We are to determine if the flow is rotational, and if so calculate the *z*-component of vorticity.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The velocity components are given by

*Velocity components, 2-D Poiseuille flow*: $$u = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right) \qquad v = 0 \qquad (1)$$

If the vorticity is non-zero, the flow is rotational. So, we calculate the *z*-component of vorticity,

*z-component of vorticity*:
$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - \frac{1}{2\mu}\frac{dP}{dx}\left(2y - h\right) = -\frac{1}{2\mu}\frac{dP}{dx}\left(2y - h\right) \qquad (2)$$

Since vorticity is non-zero, **this flow is rotational**. Furthermore, in the lower half of the flow ($y < h/2$) the vorticity is negative (note that $dP/dx$ is negative). Thus, **particles rotate in the clockwise direction in the lower half of the flow**. Similarly, **particles rotate in the counterclockwise direction in the upper half of the flow**.

***Discussion*** The vorticity varies linearly across the channel.

## 4-83

**Solution** For the given velocity field for 2-D Poiseuille flow, we are to calculate the two-dimensional linear strain rates and the shear strain rate.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The linear strain rates in the *x* direction and in the *y* direction are

*Linear strain rates*:
$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \mathbf{0} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \mathbf{0} \tag{1}$$

The shear strain rate in the *x*-*y* plane is

*Shear strain rate*:
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{1}{2\mu}\frac{dP}{dx}(2y-h)+0\right) = \frac{1}{4\mu}\frac{dP}{dx}(2y-h) \tag{2}$$

Fluid particles in this flow have non-zero shear strain rate.

***Discussion*** Since the linear strain rates are zero, fluid particles deform (shear), but do not *stretch* in either the horizontal or vertical directions.

## 4-84

**Solution** For the 2-D Poiseuille flow velocity field we are to form the 2-D strain rate tensor and determine if the *x* and *y* axes are principal axes.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** The two-dimensional strain rate tensor, $\varepsilon_{ij}$, in the *x*-*y* plane,

*2-D strain rate tensor*:
$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} \tag{1}$$

We use the linear strain rates and the shear strain rate from the previous problem to generate the tensor,

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4\mu}\frac{dP}{dx}(2y-h) \\ \frac{1}{4\mu}\frac{dP}{dx}(2y-h) & 0 \end{pmatrix} \tag{2}$$

Note that by symmetry $\varepsilon_{yx} = \varepsilon_{xy}$. If the *x* and *y* axes were principal axes, the diagonals of $\varepsilon_{ij}$ would be non-zero, and the off-diagonals would be zero. Here we have the opposite case, so **the x and y axes are not principal axes**.

***Discussion*** The principal axes can be calculated using tensor algebra.

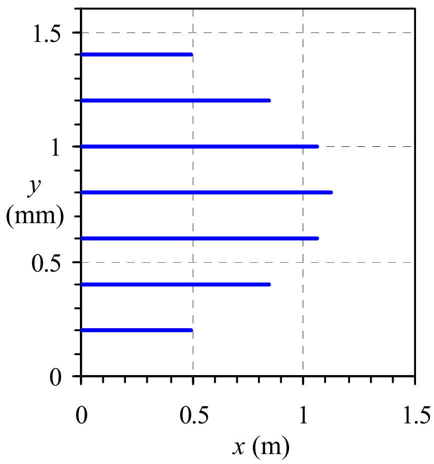


**FIGURE 1**
Pathlines for the given velocity field at $t$ = 12 s. Note that the vertical scale is greatly expanded for clarity ($x$ is in m, but $y$ is in mm).

## 4-85

**Solution** For a given velocity field we are to plot several pathlines for fluid particles released from various locations and over a specified time period.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the $x$-$y$ plane.

***Properties*** For water at 40°C, $\mu = 6.53\times10^{-4}$ kg/m·s.

***Analysis*** Since the flow is steady, pathlines, streamlines, and streaklines are all straight horizontal lines. We simply need to integrate velocity component $u$ with respect to time over the specified time period. The horizontal velocity component is

*x-velocity component, 2-D Poiseuille flow*:
$$u = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right) \qquad (1)$$

We integrate as follows:

*x position, end of pathline*:
$$x = x_{\text{start}} + \int_{t_{\text{start}}}^{t_{\text{end}}} u\,dt = 0 + \int_0^{10\text{ s}}\left(\frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)\right)dt$$
$$x = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)\left(10\text{ s}\right) \qquad (2)$$

We substitute the given values of $y$ and the values of $\mu$ and $dP/dx$ into Eq. 2 to calculate the ending $x$ position of each pathline. We plot the pathlines in **Fig. 1**.

***Discussion*** Streaklines introduced at the same locations and developed over the same time period would look identical to the pathlines of Fig. 1.

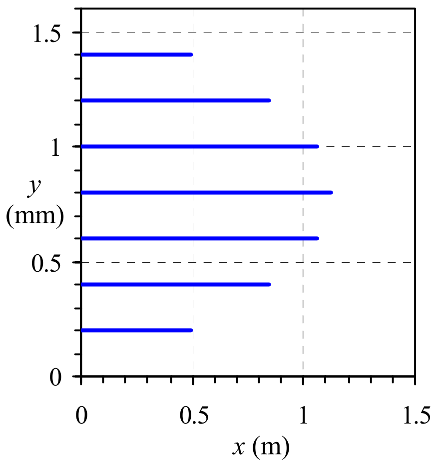


**FIGURE 1**
Streaklines for the given velocity field at $t$ = 10 s. Note that the vertical scale is greatly expanded for clarity ($x$ is in m, but $y$ is in mm).

**CD-EES 4-86**

**Solution** For a given velocity field we are to plot several streaklines at a given time for dye released from various locations over a specified time period.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the $x$-$y$ plane.

***Properties*** For water at 40°C, $\mu = 6.53\times10^{-4}$ kg/m·s.

***Analysis*** Since the flow is steady, pathlines, streamlines, and streaklines are all straight horizontal lines. We simply need to integrate velocity component $u$ with respect to time over the specified time period. The horizontal velocity component is

*x-velocity component, 2-D Poiseuille flow*:
$$u = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right) \qquad (1)$$

We integrate as follows to obtain the final $x$ location of the first dye particle released:

*x position, first dye particle of streakline*:
$$x = x_{\text{start}} + \int_{t_{\text{start}}}^{t_{\text{end}}} u\,dt = 0 + \int_0^{10\text{ s}}\left(\frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)\right)dt$$
$$x = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)\times\left(10\text{ s}\right) \qquad (2)$$

We substitute the given values of $y$ and the values of $\mu$ and $dP/dx$ into Eq. 2 to calculate the ending $x$ position of the first released dye particle of each streakline. The *last* released dye particle is at $x = x_{\text{start}} = 0$, because it hasn't had a chance to go anywhere. We connect the beginning and ending points to plot the streaklines (**Fig. 1**).

***Discussion*** These streaklines are introduced at the same locations and are developed over the same time period as the pathlines of the previous problem. They are identical since the flow is steady.

## 4-87

**Solution** For a given velocity field we are to plot several streaklines at a given time for dye released from various locations over a specified time period.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* plane.

***Properties*** For water at 40°C, $\mu = 6.53\times10^{-4}$ kg/m·s.

***Analysis*** Since the flow is steady, pathlines, streamlines, and streaklines are all straight horizontal lines. The horizontal velocity component is

*x-velocity component, 2-D Poiseuille flow*:
$$u = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right) \qquad (1)$$

In the previous problem we generated streaklines at $t$ = 10 s. Imagine the dye at the source being suddenly cut off at that time, but the streaklines are observed 2 seconds later, at $t$ = 12 s. The dye streaks will not stretch any further, but will simply move at the same horizontal speed for 2 more seconds. At each $y$ location, the $x$ locations of the first and last dye particle are thus

*x position, first dye particle of streakline*:
$$x = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)\left(12 \text{ s}\right) \qquad (2)$$

and

*x position, last dye particle of streakline*:
$$x = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)\left(2 \text{ s}\right) \qquad (3)$$

We substitute the given values of $y$ and the values of $\mu$ and $dP/dx$ into Eqs. 2 and 3 to calculate the ending and beginning $x$ positions of the first released dye particle and the last released dye particle of each streakline. We connect the beginning and ending points to plot the streaklines (**Fig. 1**).

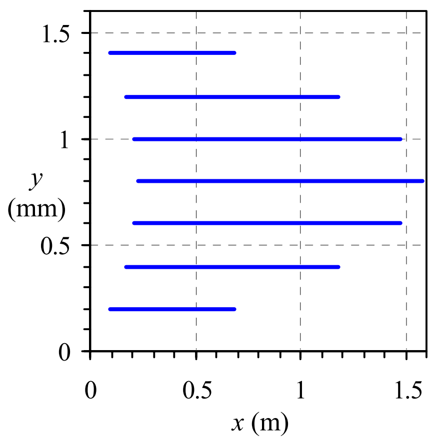


**FIGURE 1**
Streaklines for the given velocity field at $t$ = 12 s. Note that the vertical scale is greatly expanded for clarity ($x$ is in m, but $y$ is in mm).

***Discussion*** Both the left and right ends of each dye streak have moved by the same amount compared to those of the previous problem.

## 4-88

**Solution** For a given velocity field we are to compare streaklines at two different times and comment about linear strain rate in the *x* direction.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* plane.

***Properties*** For water at 40°C, $\mu = 6.53\times10^{-4}$ kg/m·s.

***Analysis*** Comparing the results of the previous two problems we see that the streaklines have not stretched at all – they have simply convected downstream. Thus, based on the fundamental definition of linear strain rate, **it is zero**:

*Linear strain rate in the x direction*:
$$\varepsilon_{xx} = 0 \qquad (1)$$

***Discussion*** Our result agrees with that of Problem 4-83.

## 4-89

**Solution** For a given velocity field we are to plot several timelines at a specified time. The timelines are created by hydrogen bubbles released from a vertical wire at $x = 0$.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x-y* plane.

***Properties*** For water at 40°C, $\mu = 6.53\times10^{-4}$ kg/m·s.

***Analysis*** Since the flow is steady, pathlines, streamlines, and streaklines are all straight horizontal lines, but *timelines* are completely different from any of the others. To simulate a timeline, we integrate velocity component *u* with respect to time over the specified time period from $t = 0$ to $t = t_{\text{end}}$. We introduce the bubbles at $x = 0$ and at many values of *y* (we used 50 in our simulation). By connecting these *x* locations with a line, we simulate a timeline. The horizontal velocity component is

*x-velocity component, 2-D Poiseuille flow*:
$$u = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right) \qquad (1)$$

We integrate as follows:

*x position of a point on the timeline at* $t_{\text{end}}$:
$$x = x_{\text{start}} + \int_{t_{\text{start}}}^{t_{\text{end}}} u\,dt = 0 + \int_0^{t_{\text{end}}} \left(\frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)\right)dt \quad \rightarrow \quad x = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - hy\right)t_{\text{end}}$$

We substitute the values of *y* and the values of $\mu$ and $dP/dx$ into the above equation to calculate the ending *x* position of each point in the timeline. We repeat for the five values of $t_{\text{end}}$. We plot the timelines in **Fig. 1**.

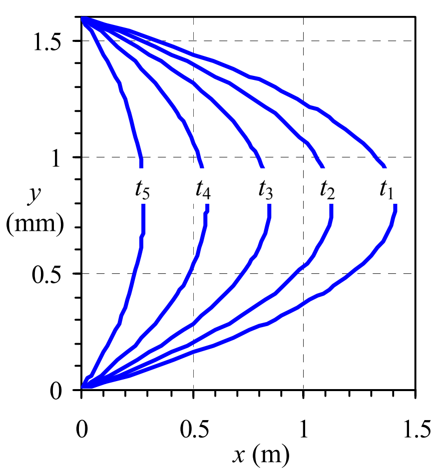


**FIGURE 1**
Timelines for the given velocity field at $t$ = 12.5 s, generated by a simulated hydrogen bubble wire at $x = 0$. Timelines created at $t_5$ = 10.0 s, $t_4$ = 7.5 s, $t_3$ = 5.0 s, $t_2$ = 2.5 s, and $t_1$ = 0 s. Note that the vertical scale is greatly expanded for clarity (*x* is in m, but *y* is in mm).

***Discussion*** Each timeline has the exact shape of the velocity profile.

## 4-90

**Solution** We are to determine if the flow is rotational, and if so calculate the $\theta$-component of vorticity.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric about the $x$ axis.

***Analysis*** The velocity components are given by

*Velocity components, axisymmetric Poiseuille flow*:

$$u = \frac{1}{4\mu}\frac{dP}{dx}\left(r^2 - R^2\right) \qquad u_r = 0 \qquad u_\theta = 0 \tag{1}$$

If the vorticity is non-zero, the flow is rotational. So, we calculate the $\theta$-component of vorticity,

*$\theta$-component of vorticity*:

$$\zeta_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u}{\partial r} = 0 - \frac{1}{4\mu}\frac{dP}{dx}2r = -\frac{r}{2\mu}\frac{dP}{dx} \tag{2}$$

Since the vorticity is non-zero, **this flow is rotational**. The vorticity is positive since $dP/dx$ is negative. In this coordinate system, positive vorticity is counterclockwise with respect to the positive $\theta$ direction. This agrees with our intuition since in the top half of the flow, $\theta$ points out of the page, and the rotation is counterclockwise. Similarly, in the bottom half of the flow, $\theta$ points into the page, and the rotation is clockwise.

***Discussion*** The vorticity varies linearly across the pipe from zero at the centerline to a maximum at the pipe wall.

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## 4-91

**Solution** For the given velocity field for axisymmetric Poiseuille flow, we are to calculate the linear strain rates and the shear strain rate.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric about the $x$ axis.

***Analysis*** The linear strain rates in the $x$ direction and in the $r$ direction are

*Linear strain rates*:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \mathbf{0} \qquad \varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \mathbf{0} \tag{1}$$

Thus there is no linear strain rate in either the $x$ or the $r$ direction. The shear strain rate in the $x$-$r$ plane is

*Shear strain rate*:

$$\varepsilon_{xr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial x} + \frac{\partial u}{\partial r}\right) = \frac{1}{2}\left(0 + \frac{1}{4\mu}\frac{dP}{dx}2r\right) = \frac{r}{4\mu}\frac{dP}{dx} \tag{2}$$

Fluid particles in this flow have non-zero shear strain rate.

***Discussion*** Since the linear strain rates are zero, fluid particles deform (shear), but do not *stretch* in either the horizontal or radial directions.

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## 4-92

**Solution** For the axisymmetric Poiseuille flow velocity field we are to form the axisymmetric strain rate tensor and determine if the *x* and *r* axes are principal axes.

***Assumptions*** **1** The flow is steady. **2** The flow is incompressible. **3** The flow is axisymmetric about the *x* axis.

***Analysis*** The axisymmetric strain rate tensor, $\varepsilon_{ij}$, is

*Axisymmetric strain rate tensor*:
$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{rx} \\ \varepsilon_{xr} & \varepsilon_{xx} \end{pmatrix} \tag{1}$$

We use the linear strain rates and the shear strain rate from the previous problem to generate the tensor,

*Axisymmetric strain rate tensor*:
$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{rx} \\ \varepsilon_{xr} & \varepsilon_{xx} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{r}{4\mu}\dfrac{dP}{dx} \\ \dfrac{r}{4\mu}\dfrac{dP}{dx} & 0 \end{pmatrix} \tag{2}$$

Note that by symmetry $\varepsilon_{rx} = \varepsilon_{xr}$. If the *x* and *r* axes were principal axes, the diagonals of $\varepsilon_{ij}$ would be non-zero, and the off-diagonals would be zero. Here we have the opposite case, so **the x and r axes are not principal axes**.

***Discussion*** The principal axes can be calculated using tensor algebra.

## 4-93

**Solution** We are to determine the location of stagnation point(s) in a given velocity field.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x-y* plane.

***Analysis*** The velocity components are

*x-component of velocity*: $$u = \frac{-\dot{V}x}{\pi L}\frac{x^2+y^2+b^2}{x^4+2x^2y^2+2x^2b^2+y^4-2y^2b^2+b^4} \qquad (1)$$

and

*y-component of velocity*: $$v = \frac{-\dot{V}y}{\pi L}\frac{x^2+y^2-b^2}{x^4+2x^2y^2+2x^2b^2+y^4-2y^2b^2+b^4} \qquad (2)$$

Both $u$ and $v$ must be zero at a stagnation point. From Eq. 1, $u$ can be zero only when $x = 0$. From Eq. 2, $v$ can be zero either when $y = 0$ or when $x^2 + y^2 - b^2 = 0$. Combining the former with the result from Eq. 1, we see that **there is a stagnation point at ($x$,$y$) = (0,0), i.e. at the origin**,

*Stagnation point*: $$u = 0 \text{ and } v = 0 \text{ at } (x, y) = (0,0) \qquad (3)$$

Combining the latter with the result from Eq. 1, there appears to be another stagnation point at ($x$,$y$) = (0,$b$). However, at that location, Eq. 2 becomes

*y-component of velocity*: $$v = \frac{-\dot{V}b}{\pi L}\frac{0}{b^4-2b^2b^2+b^4} = \frac{0}{0} \qquad (4)$$

This point turns out to be a **singularity point** in the flow. Thus, the location (0,$b$) is *not* a stagnation point after all.

***Discussion*** There is only one stagnation point in this flow, and it is at the origin.

**4-94**

**Solution** We are to draw a velocity vector plot for a given velocity field.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x-y* plane.

***Analysis*** We generate an array of *x* and *y* values in the given range and calculate *u* and *v* from Eqs. 1 and 2 respectively at each location. We choose an appropriate scale factor for the vectors and then draw arrows to form the velocity vector plot (**Fig. 1**).

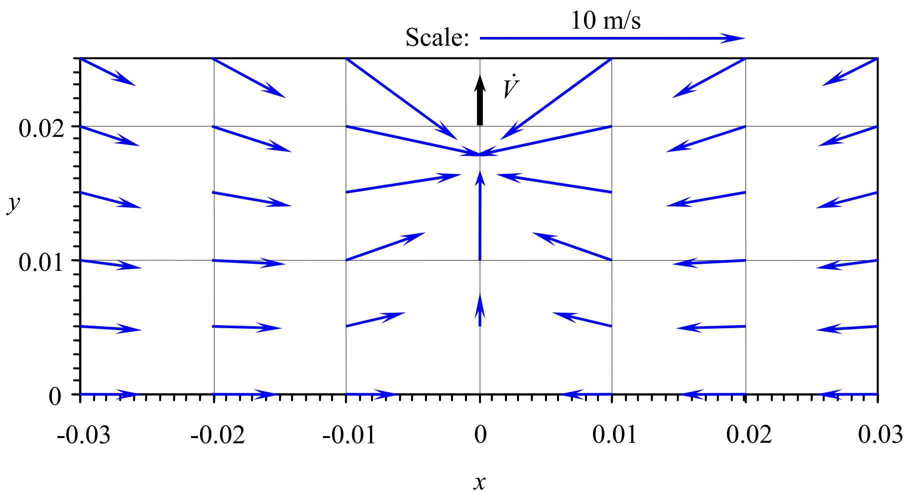


**FIGURE 1**
Velocity vector plot for the vacuum cleaner; the scale factor for the velocity vectors is shown on the legend. *x* and *y* values are in meters. The vacuum cleaner inlet is at the point *x* = 0, *y* = 0.02 m.

It is clear from the velocity vector plot how the air gets sucked into the vacuum cleaner from all directions. We also see that there is no flow through the floor.

***Discussion*** We discuss this problem in more detail in Chap. 10.

**4-95**

**Solution** We are to calculate the speed of air along the floor due to a vacuum cleaner, and find the location of maximum speed.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

***Analysis*** At the floor, $y = 0$. Setting $y = 0$ in Eq. 2 of Problem 4-93 shows that $v = 0$, as expected – no flow through the floor. Setting $y = 0$ in Eq. 1 of Problem 4-93 results in the speed along the floor,

*Speed on the floor*:

$$u = \frac{-\dot{V}x}{\pi L}\frac{x^2+b^2}{x^4+2x^2b^2+b^4} = \frac{-\dot{V}x}{\pi L}\frac{x^2+b^2}{\left(x^2+b^2\right)^2} = \frac{-\dot{V}x}{\pi L\left(x^2+b^2\right)} \tag{1}$$

We find the maximum speed be differentiating Eq. 1 and setting the result to zero,

*Maximum speed on the floor*:

$$\frac{du}{dx} = \frac{-\dot{V}}{\pi L}\left[\frac{-2x^2}{\left(x^2+b^2\right)^2} + \frac{1}{x^2+b^2}\right] = 0 \tag{2}$$

After some algebraic manipulation, we find that Eq. 2 has solutions at $x = b$ and $x = -b$. **It is at $x = b$ and $x = -b$ where we expect the best performance**. At the origin, directly below the vacuum cleaner inlet, the flow is stagnant. Thus, despite our intuition, **the vacuum cleaner will work poorly directly below the inlet**.

***Discussion*** Try some experiments at home to verify these results!

**4-96**

**Solution** For a given expression for $u$, we are to find an expression for $v$ such that the flow field is incompressible.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $x$-$y$ plane.

***Analysis*** The $x$-component of velocity is given as

*x-component of velocity*:
$$u = a + b\left(x - c\right)^2 \qquad (1)$$

In order for the flow field to be incompressible, the volumetric strain rate must be zero,

*Volumetric strain rate*:
$$\frac{1}{\cancel{V}}\frac{D\cancel{V}}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{\text{Two-D}} = 0 \qquad (2)$$

This gives us a necessary condition for $v$,

*Necessary condition for v*:
$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \qquad (3)$$

We substitute Eq. 1 into Eq. 3 and integrate to solve for $v$,

*Expression for v*:
$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2b\left(x - c\right)$$
$$v = \int \frac{\partial v}{\partial y} dy = \int \left(-2b\left(x - c\right)\right) dy + f(x)$$

Note that we must add an arbitrary function of $x$ rather than a simple constant of integration since this is a partial integration with respect to $y$. $v$ is a function of both $x$ and $y$. The result of the integration is

*Expression for v*:
$$\boxed{v = -2b\left(x - c\right)y + f(x)} \qquad (4)$$

***Discussion*** We verify by plugging Eqs. 1 and 4 into Eq. 2,

*Volumetric strain rate*:
$$\frac{1}{\cancel{V}}\frac{D\cancel{V}}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2b\left(x - c\right) - 2b\left(x - c\right) = 0 \qquad (5)$$

Since the volumetric strain rate is zero for any function $f(x)$, Eqs. 1 and 4 represent an incompressible flow field.

## 4-97

**Solution** For a given velocity field we are to determine if the flow is rotational or irrotational.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis*** The velocity components are

*Velocity components for flow over a circular cylinder of radius a*:

$$u_r = V\cos\theta\left(1-\frac{a^2}{r^2}\right) \qquad u_\theta = -V\sin\theta\left(1+\frac{a^2}{r^2}\right) \tag{1}$$

Since the flow is assumed to be two-dimensional in the $r$-$\theta$ plane, the only non-zero component of vorticity is in the $z$ direction. In cylindrical coordinates,

*Vorticity component in the z direction*:

$$\zeta_z = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right) \tag{2}$$

We plug in the velocity components of Eq. 1 into Eq. 2 to solve for $\zeta_z$,

$$\begin{aligned}\zeta_z &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-V\sin\theta\left(r+\frac{a^2}{r}\right)\right)+V\sin\theta\left(1-\frac{a^2}{r^2}\right)\right]\\ &= \frac{1}{r}\left[-V\sin\theta+V\frac{a^2}{r^2}\sin\theta+V\sin\theta-V\frac{a^2}{r^2}\sin\theta\right]=0\end{aligned} \tag{3}$$

Hence, since the vorticity is everywhere zero, **this flow is irrotational**.

***Discussion*** Fluid particles distort as they flow around the cylinder, but their net rotation is zero.

**4-98**

**Solution** For a given velocity field we are to find the location of the stagnation point.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis*** The stagnation point occurs when both components of velocity are zero. We set $u_r = 0$ and $u_\theta = 0$ in Eq. 1 of Problem 4-97,

*Stagnation point*:
$$u_r = V\cos\theta\left(1-\frac{a^2}{r^2}\right)=0 \qquad \text{Either } \cos\theta = 0 \text{ or } r^2 = a^2$$
$$u_\theta = -V\sin\theta\left(1+\frac{a^2}{r^2}\right) = 0 \qquad \text{Either } \sin\theta = 0 \text{ or } r^2 = -a^2 \tag{1}$$

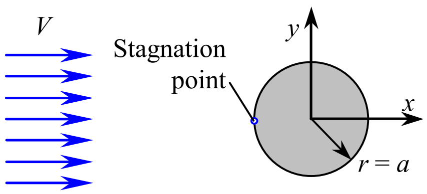


**FIGURE 1**
The stagnation point on the upstream half of the flow field is located at the nose of the cylinder at $r = a$ and $\theta = 180^\circ$.

The second part of the $u_\theta$ condition in Eq. 1 is obviously impossible since cylinder radius $a$ is a real number. Thus $\sin\theta = 0$, which means that $\theta = 0^\circ$ or $180^\circ$. We are restricted to the left half of the flow ($x < 0$); therefore we choose $\theta = 180^\circ$. Now we look at the $u_r$ condition in Eq. 1. At $\theta = 180^\circ$, $\cos\theta = -1$, and thus we conclude that $r$ must equal $a$. Summarizing,

*Stagnation point*:
$$r = a \qquad \theta = -180^\circ \tag{2}$$

Or, in Cartesian coordinates,

*Stagnation point*:
$$x = -a \qquad y = 0 \tag{3}$$

The stagnation point is located at the nose of the cylinder (Fig. 1).

***Discussion*** This result agrees with our intuition, since the fluid must divert around the cylinder at the nose.

## 4-99

**Solution** For a given stream function we are to generate an equation for streamlines, and then plot several streamlines in the upstream half of the flow field.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis***
($a$) The stream function is

*Stream function for flow over a circular cylinder*: $$\psi = V \sin\theta \left( r - \frac{a^2}{r} \right) \tag{1}$$

First we multiply both sides of Eq. 1 by $r$, and then solve the quadratic equation for $r$ using the quadratic rule. This gives us an equation for $r$ as a function of $\theta$, with $\psi$, $a$, and $V$ as parameters,

*Equation for a streamline*: $$r = \frac{\psi \pm \sqrt{\psi^2 + 4a^2V^2 \sin^2\theta}}{2V \sin\theta} \tag{2}$$

($b$) For the particular case in which $V$ = 1.00 m/s and cylinder radius $a$ = 10.0 cm, we choose various values of $\psi$ in Eq. 2, and plot streamlines in the upstream half of the flow (**Fig. 1**). Each value of $\psi$ corresponds to a unique streamline.

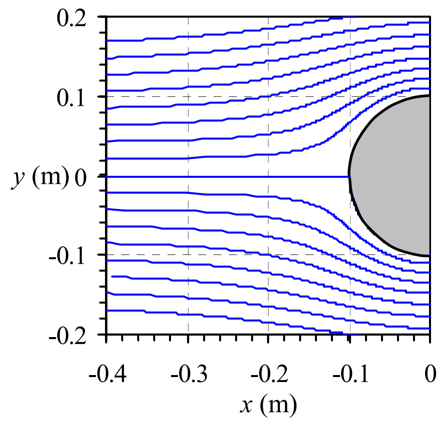


**FIGURE 1**
Streamlines corresponding to flow over a circular cylinder. Only the upstream half of the flow field is plotted.

***Discussion*** The stream function is discussed in greater detail in Chap. 9.

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## 4-100

**Solution** For a given velocity field we are to calculate the linear strain rates $\varepsilon_{rr}$ and $\varepsilon_{\theta\theta}$ in the $r$-$\theta$ plane.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis*** We substitute the equation of Problem 4-97 into that of Problem 4-91,

*Linear strain rate in r direction*: $$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = 2V\cos\theta \frac{a^2}{r^3} \tag{1}$$

and

*Linear strain rate in $\theta$ direction*:
$$\varepsilon_{\theta\theta} = \frac{1}{r}\left[\frac{\partial u_\theta}{\partial \theta} + u_r\right] = \frac{1}{r}\left[-V\cos\theta\left(1+\frac{a^2}{r^2}\right) + V\cos\theta\left(1-\frac{a^2}{r^2}\right)\right] = -2V\cos\theta\frac{a^2}{r^3} \tag{2}$$

The linear strain rates are non-zero, implying that fluid line segments *do* stretch (or shrink) as they move about in the flow field.

***Discussion*** The linear strain rates decrease rapidly with distance from the cylinder.

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**4-101**

**Solution** We are to discuss whether the flow field of the previous problem is incompressible or compressible.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis*** For two-dimensional flow we know that a flow is incompressible if its volumetric strain rate is zero. In that case,

*Volumetric strain rate, incompressible 2-D flow in the x-y plane*:

$$\frac{1}{\cancel{V}}\frac{D\cancel{V}}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

We can extend Eq. 1 to cylindrical coordinates by writing

*Volumetric strain rate, incompressible 2-D flow in the r-θ plane*:

$$\frac{1}{\cancel{V}}\frac{D\cancel{V}}{Dt} = \varepsilon_{rr} + \varepsilon_{r\theta} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\left[\frac{\partial u_\theta}{\partial \theta} + u_r\right] = 0 \qquad (2)$$

Plugging in the results of the previous problem we see that

*Volumetric strain rate for flow over a circular cylinder*:

$$\frac{1}{\cancel{V}}\frac{D\cancel{V}}{Dt} = 2V\cos\theta\frac{a^2}{r^3} - 2V\cos\theta\frac{a^2}{r^3} = 0 \qquad (3)$$

Since the volumetric strain rate is zero everywhere, **the flow is incompressible**.

***Discussion*** In Chap. 9 we show that Eq. 2 can be obtained from the differential equation for conservation of mass.

## 4-102

**Solution** For a given velocity field we are to calculate the shear strain rate $\varepsilon_{r\theta}$.

***Assumptions*** **1** The flow is steady. **2** The flow is two-dimensional in the $r$-$\theta$ plane.

***Analysis*** We substitute the equation of Problem 4-97 into that of Problem 4-91,

*Shear strain rate in r-θ plane*:

$$\varepsilon_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

$$= \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(-\frac{V\sin\theta}{r} - V\sin\theta\frac{a^2}{r^3}\right) + \frac{1}{r}\left(-V\sin\theta\left(1-\frac{a^2}{r^2}\right)\right)\right] \tag{1}$$

$$= \frac{1}{2}V\sin\theta\left[\frac{1}{r} + 3\frac{a^2}{r^3} - \frac{1}{r} + \frac{a^2}{r^3}\right] = 2V\sin\theta\frac{a^2}{r^3}$$

which reduces to

*Shear strain rate in r-θ plane*:

$$\boxed{\varepsilon_{r\theta} = 2V\sin\theta\frac{a^2}{r^3}} \tag{2}$$

The shear strain rate is non-zero, implying that fluid line segments *do* deform with shear as they move about in the flow field.

***Discussion*** The shear strain rate decreases rapidly (as $r^{-3}$) with distance from the cylinder.